Advanced Macroeconomics

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Specifying a Model

- In general, a model in macroeconomics must specify the following ingredients:
  - Households: These have preferences over allocations. Usually, we assume they seek the allocation that maximizes their utility subject to a budget constraint.
  - Firms: They seek to maximize profits subject to feasibility and their technology constraints.
  - Government: They instrument policies subject to a government constraint.
  - Other Market Participants: Banks, Entrepreneurs, Intermediate Inputs Producers, etc...
  - Information.
  - Equilibrium Concept.

Which is the Best Model?

- Without being more specific? **Absolutely no one!**
- You can only determine if a model is useful in the context of a question you want to answer.
- Are more belts and whistles always better?
  - No!
  - We introduce a new feature to a model if it is really necessary in the light of the question we are trying to answer.
- Models are like maps.

A Simple Endowment Economy

- Consider an economy where there is a mass of identical individuals with preferences given by:

$$u(c) = \sum_{t=0}^{\infty} \beta^t log(c_t),$$

where $c = (c_0, c_1, c_2, ...)$ is a consumption allocation in every period. We assume $\beta \in (0, 1)$.

- Consumers are endowed with a sequence of consumption units each period $e = (e_0, e_1, e_2, ...)$, given by

$$e_t = 2 \text{ for every } t,$$

and every consumer has the same endowment.

- In this economy there are no firms or government.

A Simple Endowment Economy: Market Structure

- Throughout the course, we will study two types of **competitive** market structures.

- **Arrow-Debreu Markets**: In this class of markets, consumers meet at period $t = 0$ and trade **all** commodities.
  - At $t \geq 1$ consumption is delivered according to what agents agreed at $t = 0$.
  - There is no trade after the initial period.

- **Sequential Markets**: When markets are sequential, we allow the possibility for trade at every period.
  - There are one-period bonds that allow consumers to either borrow or save part of their endowment for following periods.
  - Agents can only borrow until a certain limit and must always repay their debts.

Arrow-Debreu Markets

- In an Arrow-Debreu setting, households face each period a price p_t which captures the price, in terms of period $t = 0$ goods, of one unit of consumption that will be delivered at period t.

- Hence, a consumer faces the following budget constraint:

$$\sum_{t=0}^{\infty} p_t c_t \leq \sum_{t=0}^{\infty} p_t e_t.$$

- Consequently, each consumer solves the following problem:

$$\max_{c_t} \sum_{t=0}^{\infty} \beta^t log(c_t), \text{ subject to}$$

$$\sum_{t=0}^{\infty} p_t c_t \leq \sum_{t=0}^{\infty} p_t e_t$$

$$c_t \geq 0$$

Arrow-Debreu Equilibrium

- As it is standard, the optimal consumption sequence is a function of prices.
  - **Consumers are not concerned about feasibility, only about demanding according to the prices they see**.
- Hence, we seek the "right" sequence of prices such that, when consumers face them they demand exactly what is available in the economy.
- A **competitive Arrow-Debreu equilibrium** are prices $\{\hat{p}_t\}_{t=0}^{\infty}$ and an allocation $\{\hat{c}_t\}_{t=0}^{\infty}$ such that:
  1. Given prices $\{\hat{p}_t\}_{t=0}^{\infty}$ consumers solve:

$$\max_{c_t} \sum_{t=0}^{\infty} \beta^t log(c_t), \text{ subject to}$$

$$\sum_{t=0}^{\infty} \hat{p}_t c_t \leq \sum_{t=0}^{\infty} \hat{p}_t e_t$$

$$c_t \geq 0$$

  2. Each period markets clear: $c_t = e_t$.

Solving For An Arrow-Debreu Equilibrium

- The Lagrangian of the problem is:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t log(c_t) + \lambda \left[\sum_{t=0}^{\infty} \hat{p}_t e_t - \hat{p}_t c_t \right].$$

- Hence, the First Order Conditions (FOCs) are:

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 \Rightarrow \ \frac{\beta^t}{c_t} = \lambda p_t,$$
$$\frac{\partial \mathcal{L}}{\partial c_{t+1}} = 0 \Rightarrow \ \frac{\beta^{t+1}}{c_{t+1}} = \lambda p_{t+1}.$$

- Combining these equations, we attain:

$$\frac{p_t}{p_{t+1}} = \frac{c_{t+1}}{\beta c_t}.$$

Solving For An Arrow-Debreu Equilibrium

- Now, we use the second fundamental ingredient of an Arrow-Debreu equilibrium: market clearing. Since $c_t = e_t = 2$ at every period, then:

$$\frac{p_t}{p_{t+1}} = \frac{c_{t+1}}{\beta c_t} = \frac{1}{\beta}.$$

- In an Arrow-Debreu setting, we can always normalize one price to one (why?). Hence, we normalize $p_0 = 1$. Therefore, the sequence of equilibrium prices is given by:

$$p_0 = 1$$
$$p_1 = \beta$$
$$p_2 = \beta^2$$
$$p_3 = \beta^3 ...$$

Pareto Optimality and First Welfare Theorem

- Is the Arrow-Debreu market structure "efficient"?
- Throughout the course, we will be dealing with the notion of **Pareto Optimality** as a concept of efficiency:
  - We say a consumption allocation is **feasible** if $c_t \geq 0$ and $c_t \leq e_t$ for all t.
  - A consumption allocation $\{c_t\}_{t=0}^{\infty}$ is **Pareto Efficient** if there is no other feasible allocation $\{\tilde{c}_t\}_{t=0}^{\infty}$ such that:[1]

$$u(\tilde{c}) > u(c).$$

- **First Welfare Theorem**: Let $\{c_t\}_{t=0}^{\infty}$ be a competitive Arrow-Debreu equilibrium allocation. Then, it is Pareto Efficient.

[1] Notice that, since we are assuming a representative consumer, this is the Pareto Efficiency definition when we only have one consumer. How do we define it if we have multiple consumers?

First Welfare Theorem Implications

- The First Welfare Theorem is a very useful and general result.
  - It holds under very mild assumptions such as (weak) monotonicity of preferences.
- This theorem **does not** say that every Pareto Efficient allocation is a result of a competitive Arrow-Debreu equilibrium.
  - Usually, when we have a Pareto Efficient allocation we still need to find the "right" price sequence that supports that allocation as a competitive equilibrium outcome.
  - This can be done usually, but there are exceptions.
  - Second Welfare Theorem.
- Can you think of an example of a Pareto Efficient allocation that is not an allocation resulting from a competitive equilibrium?

Sequential Markets

- In a sequential markets structure, the consumers face a budget constraint each period.
- Agents can use their endowment e_t either to consume c_t or to sell/buy bonds a_{t+1}.
  - If a consumer buys a_{t+1} units of the bond at period t, she will receive a_{t+1} units of the consumption good in period $t+1$.
- Let r_{t+1} denote the interest rate of such bonds between period t and $t+1$.
- Hence, the household's budget constraint at period t is given by:

$$c_t + \frac{a_{t+1}}{1 + r_{t+1}} = e_t + a_t.$$

- We interpret $q_t = 1/(1 + r_{t+1})$ as the price of one unit of consumption in $t+1$ in terms of goods at $t = 0$.

Sequential Markets Equilibrium

- A **Competitive Sequential Markets Equilibrium** is characterized by an allocation $\{c_t, a_{t+1}\}_{t=0}^{\infty}$ and interest rates $\{\hat{r}_{t+1}\}_{t=0}^{\infty}$ such that:
  1. Given interest rates $\{\hat{r}_{t+1}\}_{t=0}^{\infty}$, the allocation $\{c_t, a_{t+1}\}_{t=0}^{\infty}$ solves:

$$\max_{\{c_t, a_{t+1}\}} \quad \sum_{t=0}^{\infty} \beta^t log(c_t) \text{ subject to}$$

$$c_t + \frac{a_{t+1}}{1 + r_{t+1}} = e_t + a_t,$$

$$c_t \geq 0,$$

$$a_{t+1} \geq -\overline{A}.$$

  2. Each period markets clear:

$$c_t = e_t,$$

$$a_{t+1} = 0.$$

Sequential Markets and Arrow-Debreu Equilibria

- Are the allocations implied by a sequential markets equilibrium different from those implied by an Arrow-Debreu one?
- In general, the answer is NO!
- Under very mild conditions the allocations implied by each equilibrium are the same. And actually there is a very simple relationship between Arrow-Debreu prices and interest rates:

$$1 + \hat{r}_{t+1} = \frac{\hat{p}_t}{\hat{p}_{t+1}}.$$

- Since this equivalence exists, then the First Welfare Theorem is also true for Sequential Markets (why?).
- Why then use two definitions? In more general models, sometimes it is more useful to use one definition over the other.

Outline

Kaldor's Facts (1957)

- Nicholas Kaldor pointed out in a now seminal article, *A Model of Economic Growth*, six facts about economic growth. Since then, these have become known in the literature as **stylized facts**:

  1. Output per worker has grown at a roughly constant rate.
  2. Capital per worker grows over time at basically the same rate.
  3. The Capital/Output ratio has been constant over time.
  4. The return of Capital has been constant.
  5. The share of Capital and Labor expenditures used in production has been constant.
  6. Real wages have increased through time.

- The Classical Growth Model, which we now study, has become fairly popular since it accounts for all these facts.

US' GDP Evolution

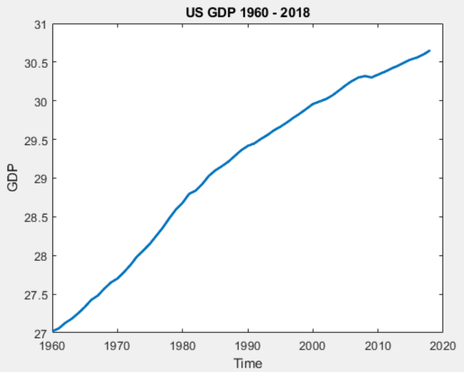

Figure: US' GDP Since 1960.

Mexico's GDP Evolution

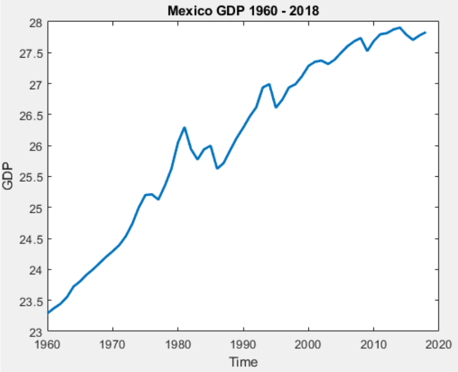

Figure: Mexico's GDP Since 1960.

The Classical Growth Model

- The basic ingredients of the Classical Growth Model are the following:
  - Households: They seek a consumption allocation c to maximize their utility. They provide labor n_t and are owners of capital k_t, receiving from both an income each period. Finally, households do investment decisions on future capital.
  - Firms: Profit maximizers who rent both labor and capital to households. They have a constant returns to scale technology.
  - Information: There is no risk in this economy, all sequences are deterministic.
  - Competitive Equilibrium Concept: Both household and firms **cannot** influence prices, they take them as given.

The Classical Growth Model: Firms

- This model is populated with a mass of identical firms. Each one faces the following production function:

$$Y_t = F(K_t, N_t) = AK_t^{\alpha} N_t^{1-\alpha},$$

which is a Cobb-Douglas production technology.

- Firms must pay a wage w_t for each unit of labor they hire and a rent R_t for each unit of capital utilized. We normalize the price of the consumption good to one.

- Hence, firms solve the following problem:

$$\max_{\{N_t, K_t\}} AK_t^{\alpha} N_t^{1-\alpha} - w_t N_t - R_t K_t \quad \text{subject to}$$

$$N_t \geq 0,$$

$$K_t \geq 0.$$

The Classical Growth Model: Firms

- The FOCs of this problem imply:

$$w_t = F_N(K_t, N_t) = (1-\alpha)A\left[\frac{K_t}{N_t}\right]^{\alpha},$$

$$R_t = F_K(K_t, N_t) = \alpha A\left[\frac{N_t}{K_t}\right]^{1-\alpha}.$$

- Notice that the share of a firm's expenses in labor is given then by:

$$w_t N_t = (1-\alpha)A\left[\frac{K_t}{N_t}\right]^{\alpha} N_t = (1-\alpha)AK_t^{\alpha}N_t^{1-\alpha} = (1-\alpha)Y_t.$$

- Hence, this implies that the firm's expenses in labor relative to output is constant:

$$\frac{w_t N_t}{Y_t} = 1-\alpha,$$

which is a Kaldor fact! Similarly for capital.

The Classical Growth Model: Households

- Households seek to maximize their utility, which depends on consumption and labor. They own the capital stock of the economy and decide how much to invest considering depreciation.

- Labor is perfectly movable within a period. However, capital is not. Households then need to decide the stock of capital they wish to have in $t+1$ during period t. They take as given the initial capital stock k_0.

- They solve the following maximization problem:

$$\max_{\{c_t, n_t, k_{t+1}, i_t\}} \sum_{t=0}^{\infty} \beta^t \left[log(c_t) - \eta \frac{n_t^{1+\nu}}{1+\nu} \right] \quad \text{subject to}$$

$$c_t + i_t = w_t n_t + R_t k_t,$$

$$i_t = k_{t+1} - (1-\delta) k_t,$$

$$c_t, \quad i_t, \quad n_t, \quad k_{t+1} \geq 0,$$

$$k_0 \ \text{ given.}$$

The Classical Growth Model: Households

- This problem can be rewritten as follows:

$$\max_{\{c_t, n_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \left[log(c_t) - \eta \frac{n_t^{1+\nu}}{1+\nu} \right] \quad \text{subject to}$$

$$c_t + k_{t+1} = w_t n_t + R_t k_t + (1-\delta) k_t$$

$$c_t, \quad k_{t+1} \geq 0,$$

$$k_0 \text{ given},$$

for which the Lagrangian is given by:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left[log(c_t) - \eta \frac{n_t^{1+\nu}}{1+\nu} \right] + \sum_{t=0}^{\infty} \lambda_t [w_t n_t + R_t k_t + (1-\delta) k_t - c_t - k_{t+1}]$$

$$= ... + \lambda_t [w_t n_t + R_t k_t + (1-\delta) k_t - c_t - k_{t+1}]$$
$$+ \lambda_{t+1} [w_{t+1} n_{t+1} + R_{t+1} k_{t+1} + (1-\delta) k_{t+1} - c_{t+1} - k_{t+2}] + ...$$

The Classical Growth Model: Households

- The FOCs of the household are:

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 \Rightarrow \frac{\beta^t}{c_t} = \lambda_t,$$
$$\frac{\partial \mathcal{L}}{\partial n_t} = 0 \Rightarrow \eta \beta^t n_t^{\nu} = w_t \lambda_t,$$
$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} = 0 \Rightarrow \lambda_t = [R_{t+1} + 1 - \delta]\lambda_{t+1}.$$

- These FOCs imply both the intra-temporal equation:

$$\eta c_t n_t^{\nu} = w_t,$$

and the Euler equation (inter-temporal equation):

$$\frac{c_{t+1}}{\beta c_t} = R_{t+1} + 1 - \delta.$$

The Classical Growth Model: Competitive Equilibrium

- In order to close up the model, we need to specify a notion of equilibrium.
- A **sequential markets competitive equilibrium** are prices $\{\hat{w}_t, \hat{R}_t\}_{t=0}^{\infty}$, an allocation for the household $\{\hat{c}_t, \hat{n}_t, \hat{k}_{t+1}\}_{t=0}^{\infty}$, and a demand vector for firms $\{\hat{Y}_t, \hat{N}_t, \hat{K}_{t+1}\}_{t=0}^{\infty}$ such that:
  1. Given prices $\{\hat{w}_t, \hat{R}_t\}_{t=0}^{\infty}$, the household's allocation $\{\hat{c}_t, \hat{n}_t, \hat{k}_{t+1}\}_{t=0}^{\infty}$ maximizes their utility subject to their period by period budget constraint.
  2. Given prices $\{\hat{w}_t, \hat{R}_t\}_{t=0}^{\infty}$, the firm's allocation $\{\hat{Y}_t, \hat{N}_t, \hat{K}_{t+1}\}_{t=0}^{\infty}$ maximizes its profits.
  3. Markets clear every period:

$$\hat{c}_t + \hat{k}_{t+1} = \hat{Y}_t + (1-\delta)\hat{k}_t,$$

$$\hat{n}_t = \hat{N}_t,$$

$$\hat{k}_{t+1} = \hat{K}_{t+1}.$$

Solving for an Equilibrium

- To attain the equilibrium, we need to use all the FOCs we have computed before and combine them with market clearing conditions.

- Notice we have (basically) three unknowns every period (c_t, n_t, k_{t+1}), so we need to attain three equations to solve for these variables. These equations are given by (make sure you understand where they come from):

$$\eta c_t n_t^{\nu} = (1-\alpha)A\left[\frac{k_t}{n_t}\right]^{\alpha},$$

$$\frac{c_{t+1}}{\beta c_t} = \alpha A\left[\frac{n_{t+1}}{k_{t+1}}\right]^{1-\alpha} + 1 - \delta,$$

$$c_t + k_{t+1} = A k_t^{\alpha} n_t^{1-\alpha} + (1-\delta)k_t.$$

- Problem: These equations are non-linear and we cannot get a closed form solution (but we can use a software like MATLAB to compute the solution).

Steady State of the Economy

- The next best thing we can do is to solve for the economy's **Steady State**.
- An economy is at its steady state whenever **all** variables become constant, i.e.:

$$c_t = c^\star, \quad n_t = n^\star, \quad k_{t+1} = k^\star.$$

- In a steady state, the three equations described above simplify and become:

$$\eta c n^{\nu+\alpha} = (1-\alpha)Ak^\alpha,$$

$$\frac{1}{\beta} = \alpha A \left[\frac{n}{k}\right]^{1-\alpha} + 1 - \delta,$$

$$c = Ak^\alpha n^{1-\alpha} - \delta k.$$

Steady State of the Economy

- We can solve analytically for the steady state. Actually we only need to find the value of steady state capital (since we can then use the other equations to find c, n).

- If we define $\tilde{A} = \alpha A[(1-\alpha)A]^{\frac{1}{\nu+\alpha}}$ and $\gamma = \frac{\alpha}{\nu+\alpha}$, it can be shown (please verify it) that:

$$k^{\star} = \left[\frac{\beta\alpha\tilde{A}}{1+\beta(1-\delta)}\right]^{\frac{1}{(1-\gamma)(1-\alpha)}}.$$

- What happens to the steady state capital whenever parameters change?

Converging to the Steady State

- If we are in a situation were $k_0 \neq k^{\star}$, it can be shown that over the course of time the economy will converge to its steady state.

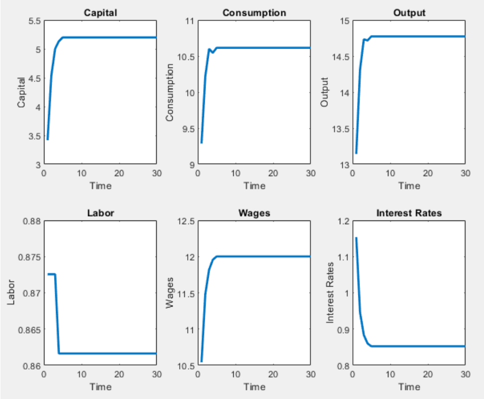

Figure: Convergence to the Steady State.

Social Planner's Problem

- We now seek to answer the following questions: is the competitive equilibrium outcome efficient? Given its technology constraints, can the economy do better?

- In order to answer both questions, we now turn to analyze what is known in the literature as the **Social Planner's Problem** (SPP).
  - The SPP captures the idea of a **centralized** economy, where a benevolent outside agent (the Social Planner) chooses/dictates the allocations that both firms and households must consume.
  - The social planner seeks to maximize welfare in the economy subject to the technology constraints that are in the economy.
  - In the context of the Classical Model, the SPP becomes a maximization problem seeking to give the household the greatest utility, subject to the feasibility constraint of the economy (that is consumption plus investment cannot exceed output).
  - **The social planner does not consider prices when optimizing, she ONLY considers feasibility**.

Social Planner's Problem

- The SPP is then:

$$\max_{\{c_t, n_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \left[log(c_t) - \eta \frac{n_t^{1+\nu}}{1+\nu} \right] \quad \text{subject to}$$

$$c_t + k_{t+1} = A k_t^{\alpha} n_t^{1-\alpha} + (1-\delta) k_t,$$

$$c_t, \quad n_t, \quad k_{t+1} \geq 0$$

$$k_0 \quad \text{given.}$$

- Notice that, almost by definition, the solution to the SPP will always be **Pareto Efficient** (why?).

Social Planner's Problem and the First Welfare Theorem

- The FOCs of the SPP are given by:

$$\frac{\beta^t}{c_t} = \lambda_t,$$

$$\eta \beta^t n_t^{\nu} = (1-\alpha) A \left[\frac{k_t}{n_t} \right]^{\alpha} \lambda_t,$$

$$\lambda_t = \left[\alpha A \left[\frac{n_{t+1}}{k_{t+1}} \right]^{1-\alpha} + 1 - \delta \right] \lambda_{t+1}.$$

- Notice that these equations yield exactly the same three equations (once we get rid of the λ's) that must hold in a competitive equilibrium!

- Hence, the **First Welfare Theorem** holds in the Classical Growth Model: any allocation that results from a competitive equilibrium is Pareto Optimal.

Towards a Recursive Representation

- We now turn to another approach to the Classical Growth Model called **Recursive Representation**.
- The recursive representation has as a goal to "simplify" the way we think of both household's and firm's problems.
  - So far, the household is looking for a sequence of variables $\{c_t, n_t, k_{t+1}\}_{t=0}^{\infty}$ that maximizes its utility.
  - These seems like a "complicated" problem, considering that this implies an infinite number of unknowns, and hence, an infinite number of equations to be solved.
  - The recursive representation simplifies this problem, since instead of solving for a **sequence** of capital, labor, and consumption, it seeks a **function** that can give us the optimal allocation in every period.

Recursive Representation of the Social Planner's Problem

- We will only focus on the recursive representation of the SPP since, as we already know, the allocations implied by this problem are the same as the ones we attain in a competitive equilibrium.

- Remember that the social planner solves its problem considering an initial k_0. Let us define $V(k_0)$ as follows:

$$V(k_0) = \max_{\{c_t, n_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \left[log(c_t) - \eta \frac{n_t^{1+\nu}}{1+\nu} \right] \quad \text{subject to}$$

$$c_t + k_{t+1} = A k_t^{\alpha} n_t^{1-\alpha} + (1-\delta) k_t,$$

$$c_t, \quad n_t, \quad k_{t+1} \geq 0,$$

$$k_0 \quad \text{given.}$$

- Hence $V(k_0)$ is the **maximal utility that the household can achieve given that the initial capital is k_0.**

Recursive Representation of the Social Planner's Problem

- Now, consider period $t = 1$. If we restrict our attention to only this period, the social planner must choose c_1, n_1 and k_2 that satisfy the FOCs **considering that k_1 is given** (it was chosen at $t = 0$).

- Hence, we can think that the problem the social planner faces at $t = 1$ is exactly the same as the one he has at $t = 0$, but considering k_1 as given instead of k_0.

- Actually, following this line of though, we can see that the problem faced by the social planner at every t is equivalent to the one he solves at $t = 0$ but taking k_t as given instead of k_0.

Recursive Representation of the Social Planner's Problem

- Considering this, let us rewrite the SPP as follows (I will not write the budget and non-negativity restrictions, but they are there):

$$V(k_0)$$

$$= \max_{\{c_t,n_t,k_{t+1}\}_{t\geq 0}} \sum_{t=0}^{\infty} \beta^t \left[log(c_t) - \eta \frac{n_t^{1+\nu}}{1+\nu} \right]$$

subject to k_0 given

$$= \max_{\{c_0,n_0,k_1\}} \left\{ log(c_0) - \eta \frac{n_0^{1+\nu}}{1+\nu} + \max_{\{c_t,n_t,k_{t+1}\}_{t\geq 1}} \sum_{t=1}^{\infty} \beta^t \left[log(c_t) - \eta \frac{n_t^{1+\nu}}{1+\nu} \right] \right\}$$

subject to k_1 given

$$= \max_{\{c_0,n_0,k_1\}} \left\{ log(c_0) - \eta \frac{n_0^{1+\nu}}{1+\nu} + \beta \ \max_{\{c_t,n_t,k_{t+1}\}_{t\geq 1}} \sum_{t=1}^{\infty} \beta^{t-1} \left[log(c_t) - \eta \frac{n_t^{1+\nu}}{1+\nu} \right] \right\}$$

subject to k_1 given

$$= \max_{\{c_0,n_0,k_1\}} \left\{ log(c_0) - \eta \frac{n_0^{1+\nu}}{1+\nu} + \beta V(k_1) \right\}.$$

Recursive Representation of the Social Planner's Problem

- Remember, this happens for **ANY** period t. Hence, we can drop out the $t = 0, 1$ indices.
- The SPP in Recursive Formulation is then:

$$V(k) = \max_{\{c,n,k'\}} \left\{ log(c) - \eta \frac{n^{1+\nu}}{1+\nu} + \beta V(k') \right\} \text{ subject to}$$

$$c + k' = Ak^{\alpha} n^{1-\alpha} + (1-\delta)k,$$

$$c, \quad n, \quad k' \geq 0.$$

- Hence the social planner is now looking for a **function** $V(\cdot)$ such that it satisfies this equation.

Some Terminology and Definitions

- The equation that the social planner now solves is known as the **Bellman Equation**.
  - The solution $V(\cdot)$ to this problem is called **Value Function**.
- We will make a distinction between two types of variables the social planner handles:
  - **State Variables**: These are all the variables that the social planner takes **as given**, but they influence the current and future decisions. In this case, there is only one state variable, k.
  - **Control Variables**: These are all the variables that, given the states, the social planner can choose to maximize her objective function. In this model there are three control variables c, n, k'.
- We will denote $g_c(k), g_n(k), g_k(k)$ the **optimal** decisions of consumption, labor, and capital given the state variable k.
  - Since $g_k(k)$ determines the value of the state in the following period, it is referred to as **Policy Function**.

SPP in Recursive Form

- Formally then, the Social Planner's Problem in Recursive Form is to find functions $V : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $g_c : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $g_n : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $g_k : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that they solve:

$$V(k) = \max_{\{c,n,k'\}} \left\{ log(c) - \eta \frac{n^{1+\nu}}{1+\nu} + \beta V(k') \right\} \text{ subject to}$$

$$c + k' = Ak^{\alpha} n^{1-\alpha} + (1-\delta)k,$$

$$c, \quad n, \quad k' \geq 0.$$

- Two things about this recursive approach:
  - We need a computer to solve it.
  - Even though it does not seem that way, it simplifies our lives to solve more complicated models.

From Recursive to Sequential

- Once we have the solution to the recursive SPP problem, we can use the policy function to recover the solution to the sequential problem.

- To do this, we only need to consider the initial k_0 and then iterate over the policy function as follows:

$$k_1 = g_k(k_0),$$
$$k_2 = g_k(k_1) = g(g(k_0)),$$
$$k_3 = g_k(k_2) = g(g(g(k_0)))...$$

- Also, we can use g_c and g_n to recover the optimal sequence of consumption and labor:

$$c_0 = g_c(k_0),$$
$$c_1 = g_c(k_1),$$
$$c_2 = g_c(k_2)...$$

Why Use the Recursive Representation?

- In more complicated frameworks, the recursive representation is the **only** way to think about a maximization problem.

- The recursive representation is very flexible to incorporate both more states and controls.

  - In the sequential formulation, more state variables usually means handling an extra "infinity" of unknowns and equations, which can be written of course, but it becomes impossible to solve the model (even with a computer).
  - On the other hand, more states in a recursive representation does not change the fact that you are looking for a function (of course it is a more complicated function, but it is still one function).

- We will study some advantages of the recursive representation during the rest of the course.

Example: An Economy with Human Capital

- Suppose that an economy is populated by two types of individuals: skilled (s) and unskilled (u) workers.

- Skilled workers own the capital in this economy and work for a firm that only hires skilled workers. The technology of such firm is given by:

$$F^s(k_t, n_t^s) = h_t^s k_t^\alpha n_t^{1-\alpha},$$

  where h_t^s is the human capital of skilled workers. Skilled workers must invest on human capital, which evolves according to:

$$h_{t+1}^s = (e_t^s)^\gamma (h_t^s)^{1-\gamma},$$

  where e_t^s is the investment skilled workers make on their own human capital.

- On the other hand, unskilled workers can only work for a firm which has technology $F^u(n_t^u) = h_t^u n_t^u$. They must also invest on their own human capital, which evolves according to:

$$h_{t+1}^u = (\theta e_t^u)^\gamma (h_t^u)^{1-\gamma},$$

  where $\theta < 1$ and e_t^u is the investment unskilled workers make on human capital.

An Economy with Human Capital

- Suppose that a social planner is looking to solve the consumption, labor, and human capital investment problem for each type of household **separately**. Each household values consumption and labor the same way:

$$u(c_t, n_t) = log(c_t) - \eta \frac{n_t^{1+\nu}}{1+\nu}.$$

- With this in mind, let us answer the following questions:

1. What are the state and control variables for the skilled household's problem?
2. State in a recursive representation the problem that the social planner solves for the skilled workers.
3. What are the state and control variables for the unskilled household's problem?
4. State in a recursive representation the problem that the social planner solves for the unskilled workers.

An Economy with Human Capital

- In order to figure out which of the variables are state and which one are controls, we must ask ourselves: in any given period, which of the variables involved in the model are taken as given (i.e. the social planner cannot affect them)?

- Since the skilled workers own the capital, and in a given period k_t is a fixed number, capital must be a state variable for these type of workers.

- Also, the human capital h_t^s is determined as a function of e_{t-1}^s and h_{t-1}^s, implying that in period t, this number is fixed.

- There are no other state variables, meaning that $c_t^s, e_t^s, n_t^s, k_{t+1}$ and h_{t+1}^s are control variables.

An Economy with Human Capital: Skilled Workers' Problem

- Let V^s be the value function for skilled workers. Then, the social planner solves the following recursive problem:

$$V^s(k, h^s) = \max_{\{c^s, e^s, n^s, k', (h^s)'\}} \left\{ log(c^s) - \eta \frac{(n^s)^{1+\nu}}{1+\nu} + \beta V(k', (h^s)') \right\}$$

$$c^s + e^s + k' = h^s k^\alpha (n^s)^{1-\alpha} + (1-\delta)k,$$

$$(h^s)' = (e^s)^\gamma (h^s)^{1-\gamma},$$

$$c^s, \quad e^s, \quad n^s, \quad k', \quad (h^s)' \geq 0.$$

An Economy with Human Capital: Unskilled Workers' Problem

- On the other hand, unskilled workers only have one state variable, which is h^u (why?).

- Let V^u be the value function for unskilled workers. Then, the social planner solves the following recursive problem:

$$V^u(h^u) = \max_{\{c^u, e^u, n^u, (h^u)'\}} \left\{ log(c^u) - \eta \frac{(n^u)^{1+\nu}}{1+\nu} + \beta V((h^s)') \right\}$$

$$c^u + e^u = h^u n^u,$$

$$(h^u)' = (\theta e^u)^\gamma (h^u)^{1-\gamma},$$

$$c^u, \quad e^u, \quad n^u, \quad (h^u)' \geq 0.$$

Thinking Critically About The Model

- What is the Classical Model useful for?
- What are the main assumptions of the Classical Model?
  - Do we believe these assumptions are realistic?
  - Which one would you criticize the most?
- What are the limitations of the Classical Model?
  - Give a concrete example of a situation where the Classical Model would not be useful.

Outline

Numerical Calculus

- Before we delve ourselves into MATLAB, we will learn some of the basic numerical calculus notions.
- When we are dealing with a computer, we need to take into account that:
  - There is a finite set of numbers that a computer can handle.
  - Nowadays computers use the **Floating-Point System**, which allows us to handle relatively small and large numbers.
  - This system uses bits (hence, it uses the binary system) to express numbers in the form $number \times base^{exponent}$.
  - Most computers use a 64-bits system (54 bits for the number and 10 for the exponent).
  - In a system with 64-bits the highest/smallest one can represent in a computer is in the order of 2^{256} and 2^{-256}, respectively.

Numerical Calculus: Machine Epsilon

- Even though in theory the computer can represent many numbers close to zero up to 2^{-256}, computers handle a **Machine Epsilon** which is a number ϵ_M such that if $0 < x < \epsilon_M$ the computer will interpret x as zero.
- In a 64-bit system $\epsilon_M = 2^{-52}$.
- We should be especially cautious when handling operations that involve numbers close to zero.
- In particular, we should avoid divisions a/b with either b very small or very large.
  - If b is very small, then the computer could consider it zero, leading to a division that is not well defined.
  - If b is very large then a/b could be interpreted by the computer as zero.
- The best practice is to avoid dividing at all.

Numerical Calculus: Matrices

- One example of an operation that is particularly hard for the computer to perform exactly is inverting a matrix.
- Remember how to invert a matrix. It uses a lot of divisions!
- Hence, this operation can cause us a lot of pain when programing in MATLAB (the MAT part stands for Matrix) since for this system most of the variables/object we will handle are matrices.
- Let us try to invert the following matrix usind MATLAB:

$$A = \begin{pmatrix} 0.01 & 0 & 0 & 0.001 \\ 0.0001 & 0.00002 & 0.3333 & 0 \\ 0 & 0 & 0 & 0.0000001 \\ 0 & 0.0001 & 0.001 & 0.000000001 \end{pmatrix}$$

Numerical Calculus: Matrices

- Lets call B the inverse of this A matrix.
- Now compute AB. What result should we attain in theory?
- Why do this occurs?
  - The matrix A is what we call a **numerical unstable** matrix.
  - Although it is invertible, it is “very close” of being non-invertible.
  - The determinant of this matrix is $det(A) = 10^{-16}$!!!
- As we will see, we will need to worry about using matrices in MATLAB, they can give us many problems!

Important MATLAB Functions: Help Function

- Our best friend in MATLAB will be the **help function**.
- Whenever you type *help functionname* this will display the documentation of functionname which will remind you how to handle functionname. Here is an example:

```
Command Window
>> help max
 max    Maximum elements of an array.
    M = max(X) is the largest element in the vector X. If X is a matrix, M
    is a row vector containing the maximum element from each column. For
    N-D arrays, max(X) operates along the first non-singleton dimension.

    When X is complex, the maximum is computed using the magnitude
    max(ABS(X)). In the case of equal magnitude elements the phase angle
    max(ANGLE(X)) is used.

    [M,I] = max(X) also returns the indices into operating dimension
    corresponding to the maximum values. If X contains more than one
    element with the maximum value, then the index of the first one
    is returned.
```

Important MATLAB Functions: Max Function

- If we consider a matrix A, the **max** function returns us a vector with two things:
  - First, the maximal number in A.
  - Second (and very important for some applications) the position in which the maximal number of A is.
- For example if $A = [3, 10, -5, 25, 7]$ then:

$$max(A) = [25, 4],$$

  since the highest number in A is 25 and this number is at the fourth position.
- What happens if A has more than one row? Then $max(A)$ returns two matrices, in the first one it puts the maximal elements of each column, and in the second it returns the position in which these elements are.

Important MATLAB Functions: fsolve Function

- **fsolve** is one of MATLAB's most important functions.
- Imagine we have a function $f : \mathbb{R}^n \to \mathbb{R}^m$ for which we want to find an $x^\star$ such that $f(x^\star) = 0$.
  - We use fsolve to compute this $x^\star$.
  - This function uses Newton's Method to attain this $x^\star$.
- To use this function, we need to give MATLAB two inputs:
  1. The function using @functionname(X) where X are the variables we want to use to get $x^\star$.
  2. An intial guess of $x^\star$ which is "close" to the actual solution.

fsolve Function: Remarks

- In order for fsolve to work you need:
  - That the initial guess you give is sufficiently close to the actual solution.
  - This poses a (big) problem since in many cases we don't have much idea about where is the solution.
  - Also, the function should not have many x's where it is the case that $f(x) = 0$, otherwise the method can converge to an x you do not want.
  - It can be the case that if the function is very flat around x, you will not find a solution.
- **Warning**: Newton's Method relies on two calculations the computer has problems with: taking derivatives and inverting matrices, so fsolve can some times give you an answer that is incorrect (in all fairness, it warns you about this).

Important MATLAB Functions: fminunc Function

- Imagine we want to solve:

$$\min \quad f(x),$$

  where x can is a vector and it is not constrained. Then, to solve this problem, we use MATLAB's fminunc function.

- To use this function we need to give MATLAB the same two arguments as in fsolve: the function's name and an initial guess.

- The same warnings I gave you about fsolve apply to fminunc.

Example: The Hodrick-Prescott Filter

- To help us fix ideas, we will study the Hodrick-Prescott Filter and program it in MATLAB.

- This filter is widely used in Macroeconomics, since it allows us to separate a time series $\{y_t\}$ between two components: its trend $\{\tau_t\}$ and cyclic component $\{c_t\}$.

- Hence, our objective is to come up with two series (trend and cyclic components) such that:

$$y_t = \tau_t + c_t \quad \text{for all } t.$$

- This allows us to analyze the long-run behavior of the economy (captured in the trend) and the short-run behavior (captured in the difference between y and the trend).

The Hodrick-Prescott Filter

- The Hodrick-Prescott Filter sets $\{\tau_t\}$ to be the sequence that solves:

$$\min_{\tau} \sum_{t=1}^{T} (y_t - \tau_t)^2 + \lambda \sum_{t=2}^{T-1} \left[(\tau_{t+1} - \tau_t) - (\tau_t - \tau_{t-1}) \right]^2 ,$$

  where:

  1. The first term penalizes the cyclical component $c_t = y_t - \tau_t$.
  2. The second term penalizes the smoothness of $\{\tau_t\}$.

- The size of λ is crucial to determine the trend of the series. The authors suggest to consider:
  - For annual data $\lambda = 6.25$.
  - For quarterly data $\lambda = 1600$.
  - For daily data $\lambda = 129,600$.

The Hodrick-Prescott Filter in the Computer

- To program the HP-Filter in MATLAB, we will need to use the *fminunc* function.
- In order to use this function we will need to program the objective function $f(\tau)$ given by:

$$f(\tau) = \sum_{t=1}^{T} (y_t - \tau_t)^2 + \lambda \sum_{t=2}^{T-1} \left[(\tau_{t+1} - \tau_t) - (\tau_t - \tau_{t-1})\right]^2 ,$$

  and we also need to provide an initial guess for τ.
- To program f we will consider the following "trick" (which will be very useful in future programs):
  - We will compute each of the two terms involved in f separately.
  - For each term we will create an auxiliary variable *aux* and compute it as (for example for the cyclical penaliation term):

$$aux = aux + (y_t - \tau_t)^2,$$

    iterating over t.

Value Function Iteration

- Imagine that we wish to solve the social planner's problem in recursive formulation:

$$V(k) = \max_{c,n,k'} \left\{ log(c) - \eta \frac{n^{1+\nu}}{1+\nu} + \beta V(k') \right\}$$

$$c + k' = Ak^{\alpha} n^{1-\alpha} + (1-\delta)k.$$

- The **Value Function Iteration Algorithm** is a tool that will allow us to solve this problem, that is, to attain the functions V as well as the policy function k'.

- Notice that we can get rid of the constraint, by substituting $c = Ak^{\alpha} n^{1-\alpha} + (1-\delta)k - k'$:

$$V(k) = \max_{n,k'} \left\{ log(Ak^{\alpha} n^{1-\alpha} + (1-\delta)k - k') - \eta \frac{n^{1+\nu}}{1+\nu} + \beta V(k') \right\},$$

$$V(k) = \max_{n,k'} \left\{ f(n,k',k) + \beta V(k') \right\}.$$

Value Function Iteration: Theory

- What is the function V we are looking for?
- Let $T : F \to F$ be defined as:

$$T(W) = \max_{n,k'} \left\{ f(n, k', k) + \beta W(k') \right\},$$

  where F is the space of all functions and $W \in F$ is a particular function.

- Hence, the solution of the SPP is a **Fixed Point of** T, since we seek for a function V such that:

$$V = T(V) = \max_{n,k'} \left\{ f(n, k', k) + \beta V(k') \right\}.$$

Value Function Iteration: Theory

- How do we compute this fixed point of T?
- Under certain conditions (which will always hold in the models we are considering) we can do the following procedure:

1. Consider **any** function V_0.
2. Compute V_1 as:

$$V_1(k) = \max_{n,k'} \left\{ f(n,k',k) + \beta V_0(k') \right\}.$$

3. Now, compute V_2 as:

$$V_2(k) = \max_{n,k'} \left\{ f(n,k',k) + \beta V_1(k') \right\}.$$

4. Continue to do this for $V_3, V_4, \ldots$
5. This procedure is called **Value Function Iteration**.

Value Function Iteration: Computational Implementation

- To perform this algorithm in the computer, we will first need to perform a **discretization** of our problem:
  - We will consider a grid K_{grid} of values between a certain K_{min} and K_{max}.
  - This grid will contain N points (we decide the size of N depending on the complexity of the problem).
  - How we compute K_{min} and K_{max}? Usually, we compute the steady state K^{ss} and use something like $K_{min} = (1/2)K^{ss}$ and $K_{max} = 2K^{ss}$.
  - We also need to construct a grid for labor. We do the same procedure as with capital.
- Hence, instead of allowing k, n to take any possible value, we restrict our attention to k being in K_{grid} and n in N_{grid}.

Value Function Iteration: Computational Implementation

- Then, the function V we are looking for is no longer any function, but we will restrict our attention to a function V such that for any k in our grid, it maximizes $f(n, k', k) + \beta V(k')$.
- How do we perform this maximization problem?
  - We evaluate the function f at every k, n in our grid.
  - Given a V, we compute:

$$Aux(k', n) = f(n, k', k) + \beta V(k'),$$

    for all k', n in the grid.
  - How do we update $V(k)$? We use MATLAB's **max** function:

$$[V(k) \;\; index] = max(Aux(k', n)),$$

    where $V(k)$ is the updated value function and index tells us where in the grid is the optimal labor and capital (policy function).

Value Function Iteration Algorithm

- Hence, the Value Function Iteration Algorithm is:

1. Choose $V_0 = zeros(N, 1)$ where N is the size of the grid.
2. As long as $||V_t - Vt - 1|| > \epsilon$ where ϵ is a very small number, then uptade V_{t-1} into V_t as follows:

$$Aux(k', n) = f(n, k', k) + \beta V_{t-1}(k'),$$

$$[V_t(k) \;\; index] = max(Aux(k', n)),$$

   where the new value of the policy function will be $g_t(k) = K_{grid}(index)$ and $n_t(k) = N_{grid}(index)$.
3. As a result of this procedure, we attain the optimal V and policy functions.

Outline

Models with Risk

- So far, we have assumed that both households and firms have **perfect foresight** about future endowments, capital stock, labor and consumption decisions, etc...

- We will relax this assumption by introducing stochastic shocks that affect agents when optimizing.
  - Importantly, we will make the assumption that agents know the stochastic process that generates the shocks that they will face.
  - Hence, there is uncertainty about the value of the shocks but not of where they come from.

Definitions and Notation

- Let $S = \{1, 2, 3, ..., N\}$ be the set of possible states of natures in which the economy can be.
  - For example if $S = \{1, 2\}$, the state $s = 1$ could imply that productivity is A_1 and whenever the state is $s = 2$ productivity is A_2 with $A_1 < A_2$.
- We denote $s^t = (s_0, s_1, s_2, ..., s_t)$ an event history of states, meaning the economy has faced the sequence of states $s_0, s_1, s_2, ..., s_t$ between $t = 0$ and t.
- The probability that the economy faces the event history s^t is denoted $\pi_t(s^t)$. For the moment, we will assume that for every event history and every t, $\pi_t(s^t) > 0$.

Event Histories Example

- Let us think that $S = \{s_1, s_2\}$. Event histories can get complicated very quickly, and they growth at an exponential rate as t becomes larger:

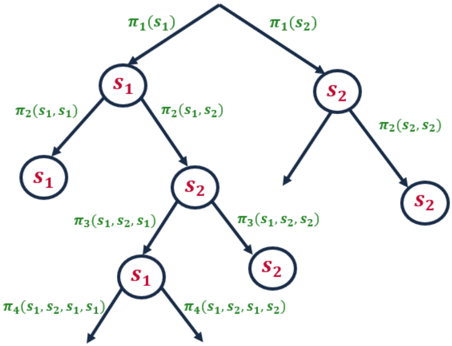

Figure: Event Histories.

Markov Processes

- One simplifying assumption that we will usually impose to π is that the process that determines shocks is **Markov**:
  - A stochastic process is **Markovian** if the realization probability of a current shock s_{t+1} only depends on the realization of s_t. That is:

$$P[s^{t+1}|s^t] = P[s_{t+1}|s^t] = P[s_{t+1}|s_0, s_1, ..., s_t] = P[s_{t+1}|s_t].$$

- Usually, we write the transition probabilities of a Markov process in matrix form:

$$Q = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix},$$

  where p_{ij} is the probability that tomorrow the shock is s_j given that today it is s_i.

An Economy with Stochastic Endowments

- To fix ideas, we will first analyze an economy where there is a representative agent, who faces uncertainty about the endowment he will have in the future.

- There are two possible states in the economy $S = \{1, 2\}$ and the endowment is given by:

$$e_t(s^t) = 1 \text{ if } s_t = 1,$$
$$e_t(s^t) = 2 \text{ if } s_t = 2,$$

- Notice that (in general) the endowment is a function of the entire event history up to period t. In this particular example, the value of the endowment only depends on the realization of s_t, however, in general realizations of shocks may depend on previous states.

Households

- Throughout the entire course, we will make the assumption that households maximize their **expected utility** from period $t = 0$ forward:

$$U(c) = \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t log(c_t) \right].$$

- From period $t = 0$'s perspective, every future c_t is stochastic, since it is a function of endowment. To be explicit about this we will denote consumption (an actually all the variables) as a function of the history of states up to period t:

$$c_t(s^t) = c(s_0, s_1, ..., s_t).$$

- Then, we can write the utility of the household as follows:

$$U(c) = \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t log(c_t) \right] = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t) log\left(c_t(s^t)\right).$$

Arrow-Debreu Competitive Equilibrium

- As before, we will define a notion of an Arrow-Debreu competitive equilibrium.
  - Remember, the key aspect of an Arrow-Debreu market structure is that prices and consumption are determined **at the initial period**.
  - In this case, prices and consumption will not only be set at $t = 0$, they will be determined before any realization of s is known.
  - Let $p_t(s^t)$ denote the price (negotiated in period $t = 0$) of one unit of consumption at period t if s^t realizes.

- Then, an Arrow-Debreu market structure (just as before) implies a **single** budget constraint for the household:

$$\sum_{t=0}^{\infty} \sum_{s^t} p_t(s^t) c_t(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t} p_t(s^t) e_t(s^t).$$

Arrow-Debreu Competitive Equilibrium Illustration

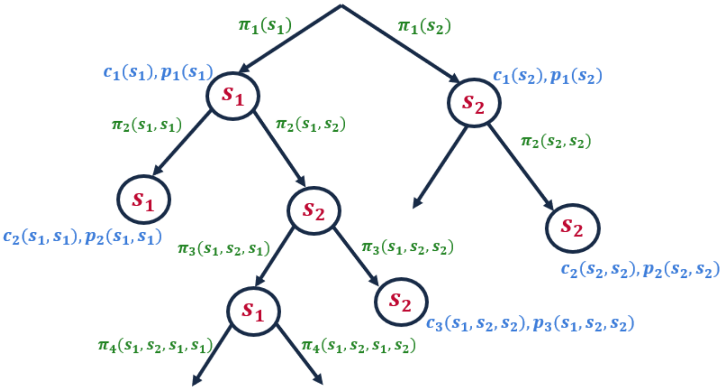

Figure: Consumption and Prices as Function of Event Histories.

Arrow-Debreu Competitive Equilibrium

- Formally an Arrow-Debreu competitive equilibrium are prices $\{(\hat{p}_t(s^t))_{s^t}\}_{t=0}^{\infty}$ and a consumption allocation $\{(\hat{c}_t(s^t))_{s^t}\}_{t=0}^{\infty}$ such that:

  1. Given prices, $\{(\hat{c}_t(s^t))_{s^t}\}_{t=0}^{\infty}$ the household solves:

$$\max_{\{c_t(s^t)\}} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t) log \left(c_t(s^t) \right) \quad \text{subject to}$$

$$\sum_{t=0}^{\infty} \sum_{s^t} p_t(s^t) c_t(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t} p_t(s^t) e_t(s^t),$$

$$c_t(s^t) \geq 0.$$

  2. Markets clear for every time and every possible state history:

$$c_t(s^t) = e_t(s^t).$$

Computing the Equilibrium

- Even though we are dealing with a heavier notation, to compute the equilibrium we need to do exactly the same thing as before: compute the FOCs and then use market clearing conditions.

- The model's Lagrangian is given by:

$$\mathcal{L} = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t) log\left(c_t(s^t)\right) + \lambda \left[\sum_{t=0}^{\infty} \sum_{s^t} p_t(s^t) e_t(s^t) - p_t(s^t) c_t(s^t) \right].$$

- Then, the FOCs are given by:

$$\frac{\partial \mathcal{L}}{\partial c_t(s^t)} = 0 \Rightarrow \frac{\beta^t \pi(s^t)}{c_t(s^t)} = \lambda p_t(s^t)$$

$$\frac{\partial \mathcal{L}}{\partial c_{t+1}(s^{t+1})} = 0 \Rightarrow \frac{\beta^{t+1} \pi(s^{t+1})}{c_t(s^{t+1})} = \lambda p_{t+1}(s^{t+1})$$

Computing the Equilibrium

- These FOCs imply a couple of interesting things.
  - Suppose $s_t = 1$ (meaning $e_t(s^t) = 1$). Then $p_t(s^t)\lambda = \beta^t \pi(s^{t-1}, 1)$.
  - Now, if $s_t = 2$ (meaning $e_t(s^t) = 2$). Then $p_t(s^t)\lambda = \beta^t \pi(s^{t-1}, 2)/2$. Therefore, the relative price of being in state 1 over state 2 given any previous history s^{t-1} is:

$$\frac{p_t(s^{t-1}, 1)}{p_t(s^{t-1}, 2)} = \frac{2\pi(s^{t-1}, 1)}{\pi(s^{t-1}, 2)},$$

    implying that the relative price between being in either state depends on the probability that the shocks shift the economy towards that way!
  - Similarly, if at time t the economy has faced s^t shocks, then the price of being at state s_{t+1} tomorrow relative to today is given by:

$$\frac{p_{t+1}(s^t, s_{t+1})}{p_t(s^t)} = \frac{\beta\pi(s^t, s_{t+1})}{\pi(s^t)} = \beta\pi(s_{t+1}|s^t).$$

Prices in General

- Notice that the FOCs we computed hold for every possible period. In particular they hold for the initial one:

$$\frac{\pi(s_0)}{c_t(s_0)} = \lambda p_0(s^0)$$

- For any other period t, the FOCs imply:

$$\frac{\beta^t \pi(s^t)}{c_t(s^t)} = \lambda p_t(s^t)$$

- Since we are in an Arrow-Debreu setting, we can always normalize one price to one (why?), so we impose $p_0(s^0) = 1$. Then, the price at any period t for any possible history of shocks is given by:

$$p_t(s^t) = \beta^t \frac{\pi(s^t)}{\pi(s_0)} \left[\frac{e_0(s_0)}{e_t(s^t)} \right],$$

hence, the price of consumption at every period is a function of the probability that we reach s^t conditional on s_0 and of how much resources are available after history s^t compared to s_0.

Efficiency?

- One natural question we must ask in the context of a model with uncertainty is: does the presence of risk leads to inefficiencies? Could the consumers do any better given that they face shocks?
- Fortunately for us, the answer is NO! And there are two main reasons:
  - First, we are assuming that households **know** the process that generates all the shocks they face.
  - In the context of our stochastic endowment model, households know $\pi(s^t)$ for every possible history of shocks.
  - Second, we are assuming **Complete Markets**, meaning that we are allowing households to buy consumption insurance for any possible history.
- Hence in the presence of complete information about the shocks and complete markets the **First Welfare Theorem** holds.

Complete/Incomplete Markets Example

- Mariana just bought a new luxury car worth $100,000$. Imagine that she faces $S = \{\text{not crash}, \text{minor bump}, \text{almost total loss}\}$.

- The probability that Mariana will not crash at period t (regardless of the past) is $\pi(\text{not crash}) = 0.5$, while $\pi(\text{mild bump}) = 0.4$.

- The utility of Mariana at state s is given by

$$U(s) = 100000 - C(s) - S(s) + R(s),$$

  where $C(s)$ is the price she must pay to fix her car at state s, $S(s)$ is the price she must pay to an insurance company if she wants to be protected at state s, and $R(s)$ is the amount of money the insurance company gives her back at s.

- Suppose that these functions take the following values:

$$C(\text{not crash}) = 0, \quad C(\text{mild bump}) = 10000, \quad C(\text{almost total loss}) = 95000,$$

$$S(\text{not crash}) = 0, \quad S(\text{mild bump}) = 30000, \quad S(\text{almost total loss}) = 45000,$$

$$R(\text{not crash}) = 0, \quad R(\text{mild bump}) = 5000, \quad R(\text{almost total loss}) = 75000,$$

Complete/Incomplete Markets Example

- Given this numbers, if Mariana decides to insure herself at any possible state, her expected utility is given by:

$$\mathbb{E}[\text{Full Insurance}] = (0.5)(100000) + (0.4)(65000) + (0.1)(35000) = 79,500.$$

- On the other hand, if she does not insure against some crash:

$$\mathbb{E}[\text{No Insurance } s_3] = (0.5)(100000) + (0.4)(65000) + (0.1)(5000) = 76,500.$$

$$\mathbb{E}[\text{No Insurance } s_2] = (0.5)(100000) + (0.4)(90000) + (0.1)(35000) = 89,500.$$

$$\mathbb{E}[\text{No Insurance}] = (0.5)(100000) + (0.4)(90000) + (0.1)(5000) = 86,500.$$

- Hence what is optimal for Mariana is to not insure against a mild crash but insure against an almost total loss.

Complete/Incomplete Markets Example

- Now imagine a scenario in which there is only one insurance available for which she needs to pay S = 25000 to acquire it, and it gives back to her R = 40000 if any accident occurs. Will Mariana be better off in this world?

- If she buys this insurance, her expected utility is

$$(0.5)(80000) + (0.4)(105000) + (0.1)(20000) = 88,000.$$

- If she does not buy the insurance, her expected utility is:

$$(0.5)(100000) + (0.4)(90000) + (0.1)(5000) = 86,500.$$

- Hence, Mariana buys the insurance but will have a lower utility in a situation where the insurance market is limited!

- This is an example of an **Incomplete Market**, in which consumers cannot fully insure against all possible future shocks.

A Final Note on Incomplete Markets

- Incomplete markets are heavily studied in the literature about risk and uncertainty.
- One important example: a model that has risk and incomplete markets can be used to rationalize wealth inequality.
  - How to link inequality and risk? Imagine a world where all consumers have an identical initial wealth. However, agents cannot fully insure themselves against future wealth shocks.
  - Hence, this market structure will benefit those that receive positive shocks and do not need to use their (limited) insurance. On the other hand, those that were unlucky will loose wealth just because of the shocks they received!
  - This class of frameworks were first studied by Hugget (1993) and Aiyagari (1994).
  - We will study another source of potential inequality: age!

Sequential Markets

- Coming back to complete markets, a household faces a sequential market structure if it can buy a one-period bond that is **state dependent**.

- Suppose the economy has faced s^t. At this moment of time, the household can buy a one period bond that will pay it only if state s_{t+1} materializes. Let $a_{t+1}(s^t, s_{t+1})$ be such bond.

- Then the period by period budget constraint of the household is given by:

$$c_t(s^t) + \sum_{s_{t+1}} q_t(s^t, s_{t+1}) a_{t+1}(s^t, s_{t+1}) \leq e_t(s^t) + a_t(s^t),$$

  where $q_t(s^t, s_{t+1})$ is the price of the bond that pays one unit of the consumption good if state s_{t+1} takes place.

Sequential Markets Equilibrium

- A sequential markets equilibrium are prices $\{(\hat{q}_t(s^t, s_{t+1}))_{s^t, s_{t+1}}\}_{t=0}^{\infty}$ and allocations $\{(\hat{c}_t(s^t, s_{t+1}), \hat{a}_t(s^t, s_{t+1}))_{s^t, s_{t+1}}\}_{t=0}^{\infty}$ such that:

  1. Given prices, the allocation $\{(\hat{c}_t(s^t, s_{t+1}), \hat{a}_t(s^t, s_{t+1}))_{s^t, s_{t+1}}\}_{t=0}^{\infty}$ maximizes the household's maximization problem subject to its period by period budget constraint.
  2. Markets clear at every t and at every possible event history:

$$c_t(s^t) = e_t(s^t),$$

$$a_t(s^t) = 0.$$

Sequential Markets and Arrow-Debreu

- Do we need to compute FOCs and use market clearing conditions to attain the sequential equilibrium?

- No! Remember there is a relationship between Arrow-Debreu and sequential markets. The allocations implied by both are exactly the same, and the prices are related as follows:

$$
\begin{aligned}
q_t(s^t, s_{t+1}) &= \frac{p_{t+1}(s^t, s_{t+1})}{p_t(s^t)} \\
&= \beta \frac{\pi(s^t, s_{t+1})}{\pi(s^t)} \frac{e_t(s^t)}{e_{t+1}(s^t, s_{t+1})} \\
&= \beta \pi(s_{t+1}|s^t) \frac{e_t(s^t)}{e_{t+1}(s^t, s_{t+1})}
\end{aligned}
$$

Asset Pricing: Definitions and Notation

- The price series $\{(\hat{q}_t(s^t, s_{t+1}))_{s^t, s_{t+1}}\}_{t=0}^{\infty}$ is highly important, since it allows us to compute the prices of any asset we want.
- An asset is a contract made between the household and "the market" for which the household buys or sells future consumption. Assets are **perfectly enforceable**, i.e. households must comply with all the assets they acquire.
- We denote d_t the **dividend** that the consumer will receive or pay after acquiring the asset. The dividend specifies the units of consumption the consumer must give or receive at every future state.
  - For example a **Risk Free Asset** guarantees that the household will receive one unit of the consumption good no matter what state materializes in the future. In this case $d_t(s^{t+1}) = 1$.
- A question we want to answer: given any asset that pays dividend d_t, how much must "the market" pay/charge for it?

Asset Pricing: General Case

- Suppose that a given asset pays a dividend d_t starting at period $t+1$ after everyone observes that until today the economy has faced shocks s^t.

- Hence, the ex-dividend (after s^t is known) price of such asset is given by:

$$P(d_t|s^t) = \sum_{\tau=t+1}^{\infty} \sum_{s^\tau} q_\tau(s^t, ..., s_\tau) d_t(s^t, ...s_\tau).$$

- Given an asset, the implied interest rate R of the asset can be deduced as follows:

$$1 + R = \frac{1}{P(d_t|s^t)}.$$

Important Assets and Their Price

- We now study some important assets and learn how to price them.
- **One Period Risk Free Asset**: This delivers one unit of the good at $t+1$ no matter what state materializes.
  - Then the dividend payed by this asset is $d_t(s^t, s_{t+1}) = 1$ for all s_{t+1} and $d_t(s^t, s_{t+1}, ..., s_\tau) = 0$.
  - Hence, the price of this asset is:

$$P^F(d_t|s^t) = \sum_{\tau=t+1}^{\infty} \sum_{s^\tau} q_\tau(s^t, ..., s_\tau) d_t(s^t, ...s_\tau)$$
$$= \sum_{s_{t+1}} q_t(s^t, s_{t+1}).$$

- The implied interest rate by this asset is called **Risk Free Interest Rate** and is denoted R^F.

Important Assets and Their Price: Lucas Tree

- **One Period Lucas Tree Asset**: This delivers the total endowment of the consumption good at $t+1$ no matter what state materializes.
  - Then the dividend payed by this asset is $d_t(s^t, s_{t+1}) = e_{t+1}(s^{t+1})$ for all s_{t+1} and $d_t(s^t, s_{t+1}, ..., s_\tau) = 0$.
  - Hence, the price of this asset is:

$$P^L(d_t|s^t) = \sum_{\tau=t+1}^{\infty} \sum_{s^\tau} q_\tau(s^t, ..., s_\tau) d_t(s^t, ...s_\tau)$$
$$= \sum_{s_{t+1}} q_t(s^t, s_{t+1}) e_{t+1}(s^{t+1}).$$

Important Assets and Their Price: Options

- **Call Option for Tomorrow**: Also known as Option to Buy tomorrow. Imagine we want to potentially buy an asset whose future price is $P^A(d_{t+1}|s^{t+1})$. The "market" offers us to buy this asset in the future at price Q.
- An option to buy, as stated by its name, leaves to the household the decision to buy or not this asset after knowing s_{t+1}.
- The price of an option to call tomorrow is then given by:

$$
\begin{aligned}
P^{Call}(d_t|s^t) &= \sum_{\tau=t+1}^{\infty} \sum_{s^\tau} q_\tau(s^t, ..., s_\tau) d_t(s^t, ...s_\tau) \\
&= \sum_{s_{t+1}} q_t(s^t, s_{t+1}) \max\{P^A(d_{t+1}|s^{t+1}) - Q, 0\}
\end{aligned}
$$

Exhaustive Example

- Imagine an endowment economy in which three states of nature are possible $S = \{N, R, C\}$ where N stands for normal times, R stands for recession, and C stands for crisis. The endowment is given by:

$$e_t(s_t = N) = 10, \quad e_t(s_t = R) = 3, \quad e_t(s_t = C) = 1.$$

- Suppose the transition between states is given by the following Markov matrix:

$$\Pi(s_{t+1}|s_t) = \begin{pmatrix} 0.90 & 0.09 & 0.01 \\ 0.20 & 0.50 & 0.30 \\ 0.01 & 0.79 & 0.20 \end{pmatrix}$$

- Suppose that the utility of consumption is given by $u(c_t) = log(c_t)$ (the utility function we have assumed always) and that the discount factor is $\beta = 0.99$.

Exhaustive Example: Questions to Answer

- Given this setting, we seek to answer these questions:

1. Suppose at $t = 0$ the economy is at a normal state. What is the price of one unit of consumption (in terms of $t = 0$ goods) delivered at period $t = 1$ at every possible state?
2. Suppose at $t = 0$ the economy is in a crisis state. What is the price of one unit of consumption (in terms of $t = 0$ goods) delivered at period $t = 1$ at every possible state? Compare with previous question.
3. Suppose at $t = 0$ the economy is at a normal state. What is the price of a risk free asset? If the economy were at a recession? Compare the implied Risk Free Interest Rates.
4. Suppose at $t = 0$ the economy is in a crisis. What is the price of an option to call of a Lucas tree bond sold to the consumer at price $Q = 2$?

Exhaustive Example: Question 1

- Question one is asking for $q_0(N, s_1)$ for every possible s_1. As we have already discussed, this price is given by:

$$q_0(N, s_1) = \beta\pi(s_1|N)\frac{e(N)}{e(s_1)}.$$

Using the value of the endowments at each state and the transition matrix, the value of this bonds is given by:

$$q_0(N, N) = (0.99)(0.90)(1) = 0.891,$$

$$q_0(N, R) = (0.99)(0.09)\frac{10}{3} = 0.297,$$

$$q_0(N, C) = (0.99)(0.01)\frac{10}{1} = 0.099.$$

Exhaustive Example: Question 2

- Question two is asking for $q_0(C, s_1)$ for every possible s_1. As we have already discussed, this price is given by:

$$q_0(C, s_1) = \beta\pi(s_1|C)\frac{e(C)}{e(s_1)}.$$

Using the value of the endowments at each state and the transition matrix, the value of this bonds is given by:

$$q_0(C, N) = (0.99)(0.01)\frac{1}{10} = 0.0009,$$

$$q_0(C, R) = (0.99)(0.79)\frac{1}{3} = 0.2607,$$

$$q_0(C, C) = (0.99)(0.20)(1) = 0.1980.$$

Exhaustive Example: Question 3

- Question three is asking for the price of a risk free asset considering the initial state is N. Since this delivers one unit of consumption no matter the state, its price is given by:

$$P^F(d_0|s_0 = N) = \sum_{s_1} q_0(N, s_1) d_0(s_1) = \sum_{s_1} q_0(N, s_1) = 1.287.$$

- This question also asks for the price of a risk free asset considering the initial state is C. Since this delivers one unit of consumption no matter the state, its price is given by:

$$P^F(d_0|s_0 = C) = \sum_{s_1} q_0(C, s_1) d_0(s_1) = \sum_{s_1} q_0(C, s_1) = 0.4596.$$

- To compute the implied risk-free interest rates of each asset, we use the fact that $1 + R^f = 1/P^F$:

$$R^F(d_0|s_0 = N) = -22.99\%,$$
$$R^F(d_0|s_0 = C) = 117.58\%.$$

- Interpretation?

Exhaustive Example: Question 4

- The first thing we need to do is to compute the price of a Lucas Tree asset conditional that in period $t = 1$ we are at state s_1. We need to do this for every state s_1. The one period bond prices are given by:

$$q_1(N, s_2 = N) = 0.8910 \quad q_1(N, s_2 = R) = 0.2970 \quad q_1(N, s_2 = C) = 0.0990,$$

$$q_1(R, s_2 = N) = 0.0594 \quad q_1(R, s_2 = R) = 0.4950 \quad q_1(R, s_2 = C) = 0.8910,$$

$$q_1(C, s_2 = N) = 0.0009 \quad q_1(C, s_2 = R) = 0.2607 \quad q_1(C, s_2 = C) = 0.1980.$$

- These prices imply the following Lucas tree asset prices for each s_1:

$$P^L(d_1|s_1 = N) = (0.8910)(10) + (0.2970)(3) + (0.0990)(1) = 9.90$$

$$P^L(d_1|s_1 = R) = (0.0594)(10) + (0.4950)(3) + (0.8910)(1) = 2.97$$

$$P^L(d_1|s_1 = C) = (0.0009)(10) + (0.2607)(3) + (0.1980)(1) = 0.9891$$

Exhaustive Example: Question 4

- Finally, the price of an option to call a Lucas Tree bond sold to the consumer at price $Q = 2$ when the initial state of the economy is $s_0 = C$ is given by:

$$\begin{aligned} P_0^{\text{Call}}(d_0|s_0 = C) &= \sum_{s_1} q_0(C, s_1)\max\{P^L(d_1|s_1) - Q, 0\} \\ &= (0.0009)\max\{9.90 - 2, 0\} + (0.2607)\max\{2.97 - 2, 0\} \\ &+ (0.1980)0.0009)\max\{0.9891 - 2, 0\} \\ &= 0.2599 \end{aligned}$$

- Interpretation? Compare this price with the price of a risk-free asset and a Lucas Tree whenever the initial state is C.

The Classical Growth Model and Uncertainty

- We now study how to introduce uncertainty to the Classical Growth Model.
- Why do this?
  - The Classical Growth Model is useful to the understand long-run behavior of an economy.
  - In the short-run we observe fluctuations around the trend of the economy, usually referred to as **Business Cycles**.
  - Adding a stochastic component to the Classical Growth Model will allow us to explain these short run fluctuations.
  - How? In the presence of shocks firms and households will respond optimally to them, generating fluctuations in consumption, labor, output, etc...
- In the literature this model is known as the **Real Business Cycle Model** (RBC).

US' GDP Trend

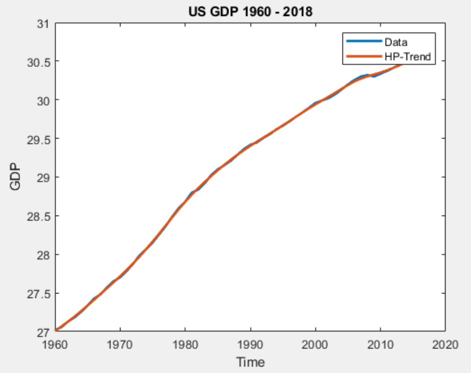

Figure: US' GDP Trend Since 1960.

US' Business Cycles

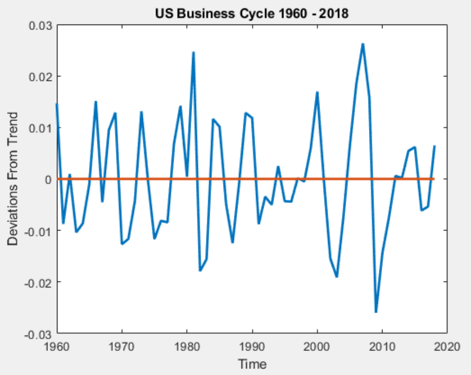

Figure: US' GDP Business Cycles Since 1960.

Mexico's GDP Trend

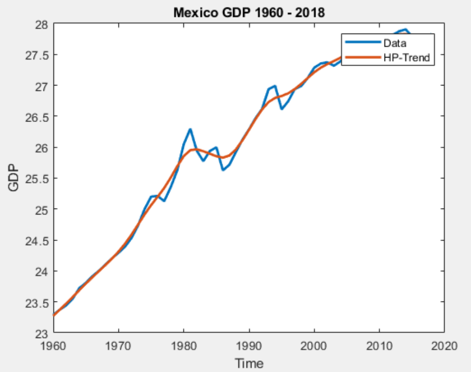

Figure: Mexico's GDP Trend Since 1960.

Mexico's Business Cycles

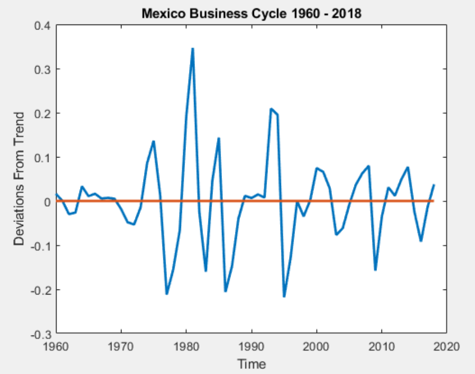

Figure: Mexico's Business Cycles Since 1960.

RBC Model

- Consider an economy where productivity z_t follows a stochastic process governed by a mean μ and variance σ^2.
  - Usually, the literature assumes that $log(z_t) \sim AR(1)$, i.e., $log(z_t) = \rho log(z_{t-1}) + \epsilon_t$ where $\epsilon_t \sim N(0, \sigma^2)$.
- Conditional on the shock, production takes place utilizing labor and capital:

$$F(K_t, N_t, z_t) = z_t K_t^{\alpha} N_t^{1-\alpha}.$$

- Then, this implies that in this model wages and the interest rate are a function of the state of the economy:

$$w_t(z_t) = (1-\alpha) z_t \left[\frac{K_t}{N_t}\right]^{\alpha},$$

$$R_t(z_t) = \alpha z_t \left[\frac{N_t}{K_t}\right]^{1-\alpha}.$$

RBC Model: Households

- Households are practically the same as in the Classical Model. The only (subtle) difference is that they have an **expected utility function**:

$$\sum_{t=0}^{\infty} \sum_{z^t} \beta^t \pi(z^t) \left[log(c_t(z^t)) - \eta \frac{n_t(z^t)^{1+\nu}}{1+\nu} \right].$$

- Households still need to invest $i_t(z^t) = k_{t+1}(z^t) - (1-\delta)k_t(z^{t-1})$ and hence their budget constraint is given by:

$$c_t(z^t) + k_{t+1}(z^t) = w_t(z^t)n_t(z^t) + R_t(z^t)k(z^{t-1}) + (1-\delta)k_t(z^{t-1}).$$

- Notice that there is a budget constraint for every t and every possible history of shocks z^t. We denote $\lambda_t(z^t)$ the Lagrange multiplier associated to this constraint.

RBC Model: Households' FOCs

- The first two FOCs are the same as before:

$$\frac{\partial \mathcal{L}}{\partial c_t(z^t)} = 0 \Rightarrow \frac{\beta^t \pi(z^t)}{c_t(z^t)} = \lambda_t(z^t),$$
$$\frac{\partial \mathcal{L}}{\partial n_t(z^t)} = 0 \Rightarrow \eta \beta^t \pi(z^t) n_t(z^t)^{\nu} = \lambda_t(z^t) w_t(z^t).$$

- We need to be more careful with the FOC respect to $k_{t+1}(z^t)$.
  - Intuitively, capital at $t+1$ is determined by the household at period t.
  - Hence **no matter what state materializes in the future** the capital stock will be fixed and will be the same across all possible z_{t+1}.
  - The FOC is then:

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}(z^t)} = 0 \quad \Rightarrow \quad \lambda_t(z^t) = \sum_{z^{t+1}} \lambda_{t+1}(z^{t+1})[R_{t+1}(z^{t+1}) + 1 - \delta].$$

RBC Model: Stochastic Euler Equation

- Combining the capital FOC with the consumption FOC we obtain the stochastic version of the Euler Equation:

$$\frac{\beta^t \pi(z^t)}{c_t(z^t)} = \sum_{z_{t+1}} \frac{\beta^{t+1}\pi(z^{t+1})}{c_{t+1}(z^{t+1})}[R_{t+1}(z^{t+1}) + 1 - \delta],$$

$$\frac{1}{c_t(z^t)} = \sum_{z_{t+1}} \frac{\beta\pi(z_{t+1}|z^t)}{c_{t+1}(z^{t+1})}[R_{t+1}(z^{t+1}) + 1 - \delta],$$

- Replacing by the equilibrium condition for the interest rate:

$$\frac{1}{c_t(z^t)} = \sum_{z_{t+1}} \frac{\beta\pi(z_{t+1}|z^t)}{c_{t+1}(z^{t+1})} \left[\alpha z_{t+1} \left[\frac{n_{t+1}(z^{t+1})}{k_{t+1}(z^t)}\right]^{1-\alpha} + 1 - \delta\right],$$

- This together with the intra-temporal equation and feasibility constraint, allows us to compute the equilibrium (of course, with the help of a computer).

RBC Model: Recursive Representation

- Dealing with an RBC model in sequential way can be a bit complicated to deal with. The recursive representation allows us to handle this model in a more natural way.
- Which are the states? Naturally, as before, capital is still a state variable. But, in an RBC there is an additional state variable: the productivity shock (why?).
- The recursive representation of the RBC model is then:

$$V(k,z) = \max_{c,n,k'} \left\{ log(c) - \eta \frac{n^{1+\nu}}{1+\nu} + \beta \sum_{z'} \pi(z'|z) V(k',z') \right\} \text{ subject to}$$

$$c + k' = zk^{\alpha} n^{1-\alpha} + (1-\delta)k,$$

$$c, \quad n, \quad k' \geq 0.$$

RBC Model: Simulation

- I now present a simulation (using MATLAB) of the RBC model.
- I will consider a productivity shock that follows an AR(1) process and will simulate the model for $T = 2000$ periods.
- All details of how to simulate the model can be found in the MATLAB folder of our course page.

RBC Model Simulation

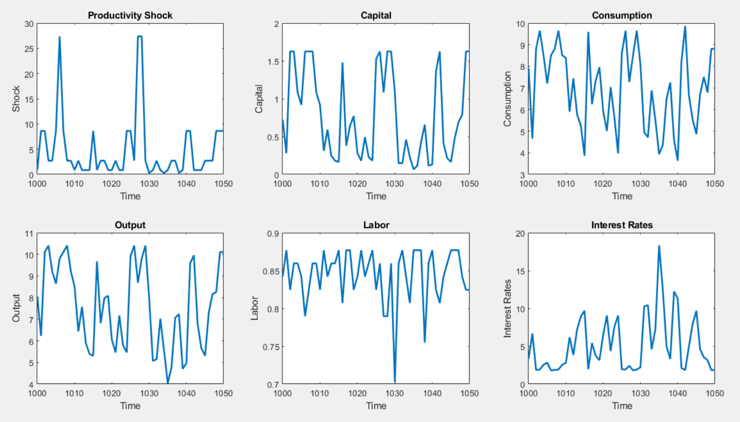

Figure: Aggregates between $t = 1000$ and $t = 1050$.

RBC Model Simulation

- Here are some statistics that will allow us to understand how this model works.

Productivity Shock Mean	5.6358
Consumption Mean	7.4140
Output Mean	8.1465
Correlation Shock and Consumption	0.0964
Correlation Shock and Output	0.2786
Correlation Shock and Labor	-0.4837
Correlation Shock and Capital	0.3481
Correlation Shock and Wages	0.3477
Correlation Shock and Interest Rate	-0.2482
Auto-correlation Consumption	0.4136
Auto-correlation Output	0.6234
Auto-correlation Capital	0.5687

Table: Statistics of the Simulated RBC Model

Takeaway of the RBC Model

- This model has some interesting predictions about the correlation of some variables over time:
  - By construction of the model, output is positively correlated with shocks (why?). However this correlation is not one, due to capital and labor decisions.
  - Interest rates do not co-move in the same direction as shocks (why?).
  - What about consumption? It is positively correlated with income (output), however, it has a smoother movement in time. Why? Because of insurance (you can easily see this in Euler Equation).
  - What about labor and wages?
- In the context of an RBC: are counter-cyclical policies (i.e. policies that go in the opposite direction of shocks) a good idea?

Thinking Critically About The Model

- What is role of uncertainty in a model?
- What are the main assumptions of the RBC Model?
  - Do we believe these assumptions are realistic?
  - Which one would you criticize the most?
- What are the limitations of the RBC Model?
  - Give a concrete example of a situation where the RBC Model would not be useful.

Further Relaxing Some Assumptions

- One (critical) assumption we have made so far is that agetns **know** the stochastic process that governs shocks.
- Let us briefly discuss how to relax this assumption.
- Two types of uncertainty:
  1. Quantifiable uncertainty: agents are unaware of future shocks but either know the distribution of such shocks or know the distribution over potential distributions of shocks.
  2. Unquantifiable uncertainty (Knightian uncertainty): agents are unaware of future shocks and do not know the distribution of such shocks.

Quantifiable Uncertainty

- Let us imagine that agents know that a distribution F governs the shocks they observe. Then, let $V_F(x)$ be the value function of the agent when he knows that F is the distribution of shocks and this agent faces some state variables x.

- Now let us assume that agents do not know the actual distribution that governs shocks but know that it has to be one of the distributions in a finite set $\mathcal{F}$. Then, the agent's value function is given by:

$$V_{\mathcal{F}}(x) = \mathbb{E}_{\mathcal{F}}\left[V_F(x)\right] = \sum_{F \in \mathcal{F}} p(F) V_F(x).$$

- Is this more realistic?

Unquantifiable Uncertainty

- Now agents does not know anything about the process that governs shocks.
- In order to be able to say something about decisions under this type of uncertainty, we need to impose a bit more structure on the model.
- We assume that the agent has a conjecture $\mathcal{M}$ of possible distributions of shocks.
- In this case, agents maximize their expected utility under the worst possible distribution in $\mathcal{M}$.
- This is what is called the **max-min criterion**, or the **robust criterion**.
- Is this more realistic?

Outline

Prices Are Sticky After All

- All the models we have studied so far assume that prices are flexible.
  - Take the RBC model. In this framework, agents respond optimally to shocks every period.
  - In particular, firms adjust their prices after observing the productivity shock.
- Are prices really that flexible?
- There has been a large literature that has empirically documented that prices are not as flexible as (for example) an RBC model assumes.
  - Prices are **sticky**, meaning they tend to last for a while before we observe a change.
  - Stickiness is the starting point of what has become known in the literature as **Neo-Keynsian Models**.

Price Trajectory Example

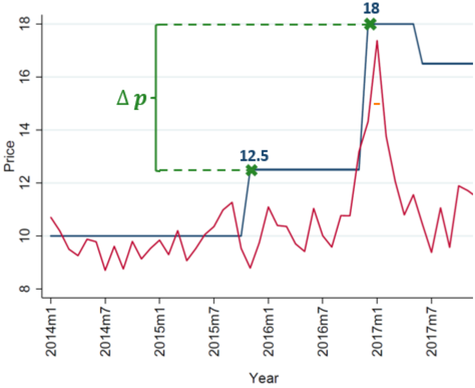

Figure: Price of a Product Through Time

Price Stickiness: Evidence From Mexico

- Before we delve ourselves into the fully Neo-Keynesian framework, we will discuss empirical evidence on price stickiness.
  - We will study the stickiness of prices and its importance in the context of the mexican economy.
  - While we do this, I will introduce some key concepts that are widely used in Neo-Keynesian models.
  - The main source of all the data I will present, is Mexico's Central Bank and some papers it has published about this topic like Capistrán, Ibarra, and Ramos-Francia (2011); Cortes (2013); Kochen (2016), among others.

Measuring Price Stickiness

- Klenow and Kryvtsov (2008) proposed to measure inflation as follows:

$$\pi_t = \sum_{s \in \Gamma_t} \omega_t^s \Delta p_t^s,$$

where Γ_t is the set of all available products at time t, Δp_t^s is the observed price change of product s at time t and ω_t^s is the weight that product receives.

- As we already discussed, not all prices change all the time, then some of these Δp_t^s are going to be zero. This allows us to further decompose inflation as follows:

$$\pi_t = \left(\sum_{s \in \Gamma_t} \omega_t^s I_t^s \right) \left(\frac{\sum_{s \in \Gamma_t} \omega_t^s \Delta p_t^s}{\sum_{s \in \Gamma_t} \omega_t^s I_t^s} \right),$$

where I_t^s is equal to one if the price of product s changed at time t. This decomposes inflation into two components:

$$\pi_t = fr_t dp_t,$$

known as **Frequency of Price Changes** (how often price change in general) and **Magnitude of Price Changes** (whenever price change, by how much they do so).

Magnitude of Price Changes in Mexico

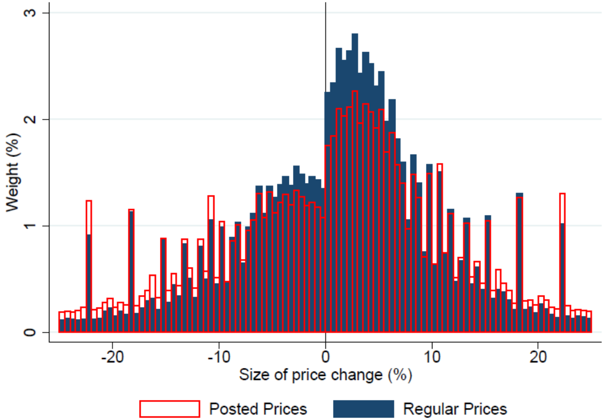

Figure: Magnitude of Price Changes (2009-2016)

Magnitude of Price Changes in Mexico

		Posted Prices						Regular Prices				
							Sales					
Major Group	Weight	dp at t	dp	$\|dp\|$	$\|dp^+\|$	$\|dp^-\|$	$\|dp\|$	dp at t	dp	$\|dp\|$	$\|dp^+\|$	$\|dp^-\|$
Processed Food	14.7	1.3	1.2	9.4	9.1	9.8	13.5	1.7	1.6	7.4	7.2	7.5
Unprocessed Food	8.4	2.3	0.5	16.0	15.3	16.6	24.0	2.4	0.5	14.0	13.3	14.7
Household Goods	2.4	1.3	0.5	11.3	10.9	11.7	15.6	1.3	1.0	7.8	7.6	8.0
Household Durables	1.7	1.5	0.6	11.7	11.2	12.2	14.3	1.9	1.5	8.1	7.8	8.6
Apparel	5.3	3.8	1.7	16.2	14.4	18.6	24.8	3.3	3.2	8.9	8.4	9.9
Transportation Goods	3.4	1.0	0.7	3.8	3.6	4.4	7.2	1.1	0.8	3.2	3.1	3.5
Recreation Goods	1.2	2.2	1.4	11.6	10.9	12.8	18.1	2.6	2.2	8.1	7.9	8.2
Health and P. Care Goods	5.3	1.2	0.5	12.5	11.9	13.1	17.5	1.2	0.9	8.1	7.7	8.5
Services	15.5	4.3	4.1	7.8	7.1	11.1	20.6	4.6	4.5	6.6	6.4	7.7
Total Sample	**57.8**	**1.2**	**1.0**	**12.4**	**11.6**	**13.5**	**18.0**	**1.6**	**1.3**	**10.1**	**9.4**	**11.1**

SOURCE: Banco de México and INEGI.
NOTES: Regular Prices denotes prices excluding sales. With the exception of dp at t, all the statistics reported were calculated for each semimonthly period and then averaged across time. dp denotes the average size of price changes. $|dp|$ is the magnitude of price changes. $|dp^+|$ and $|dp^-|$ are the magnitude of price increases and decreases, respectively. dp at t is the average size of price changes at each half-month period taking the absolute value of each period before averaging across time.

Figure: Magnitude of Price Changes Estimates (2009-2016)

Duration of Prices in Mexico

- Frequency of price changes is important because it determines duration (they are inversely related).

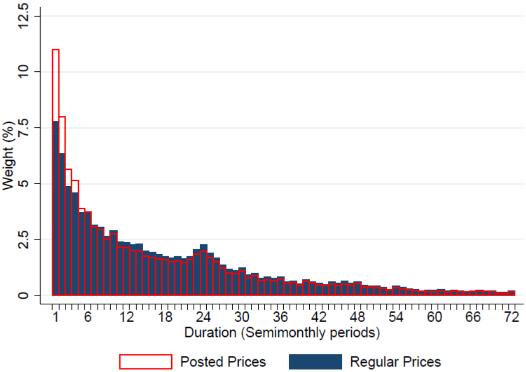

Figure: Duration of Prices (2009-2016)

Duration of Prices in Mexico

		Posted Prices				Regular Prices	
				Sales			
Major Group	Weight	*dur*	Md *dur*	*dur*	Md *dur*	*dur*	Md *dur*
Processed Food	14.7	19.3	14.0	3.7	2.0	21.5	18.0
Unprocessed Food	8.4	6.8	3.0	2.1	1.0	7.8	4.0
Household Goods	2.4	10.9	5.0	3.3	2.0	13.9	8.0
Household Durables	1.7	10.0	6.0	4.9	4.0	14.2	10.0
Apparel	5.3	23.7	18.0	6.3	4.0	29.8	23.0
Transportation Goods	3.4	8.7	6.0	5.9	5.0	9.3	7.0
Recreation Goods	1.2	23.1	17.0	6.4	4.0	25.2	19.0
Health and P. Care Goods	5.3	11.0	6.0	3.9	3.0	14.4	10.0
Services	15.5	33.3	27.0	9.5	7.0	33.7	27.0
Total Sample	**57.8**	**18.1**	**11.0**	**4.0**	**2.0**	**19.9**	**14.0**

SOURCE: Banco de México and INEGI.
NOTES: Regular Prices denotes prices excluding sales. All duration results are reported in semimonthly periods. *dur* (Md *dur*) denotes the mean (median) duration of uncensored price spells calculated as in equation (4).

Figure: Duration of Prices Estimates (2009-2016)

Neo-Keynesian Models

- The main characteristic that distinguishes a Neo-Keynesian model, is that it embodies some type of **Price Stickiness**.
  - One can assume that the prices of consumption goods are sticky.
  - But also wages or interest rates could be rigid.
- The Neo-Keynesian literature typically uses one of the following mechanisms to model price stickiness:
  - **Time Dependent Models**: This class of models assume firms set their price and can only change it if they receive some exogenous shock.
    - The **Calvo Model** is the most famous model of this type, where firms can only change their price with some probability θ.
  - **State Dependent Models**: In this type of frameworks, firms can choose what price to set, but are susceptible to shocks that may push them to not change the price.
    - The **Golosov-Lucas Menu Cost Model** is the most popular model of this class, where firms have productivity shocks every period but must pay a fixed cost to change their prices.

Krugman's Monopolistic Competition Model

- Neo-Keynesian models (NKM) typically assume that firms are the ones that generate sticky prices.
- Does it make sense to assume a representative firm?
  - Not so much, since one thing we want to capture is the **cross-section** distribution of prices.
  - If there was a representative firm, there would only be one price.
- Most of the NKM are built on the seminal Krugman model of Monopolistic Competition.
  - Here, consumers have a **taste for variety** and demand consumption of N differentiated varieties.
  - Each variety is produced by a single firm.
  - Since varieties are imperfect substitutes, firms have monopolistic power over their own variety, hence they can decide not only how much to sell, but also the price.
  - For our purposes, we will briefly discuss the simplest Monopolistic Competition model.

Krugman's Monopolistic Competition Model

- The representative consumer solves the following problem ($\sigma > 1$):

$$\max \quad \left[\sum_{i=1}^{N} c_i^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} \quad \text{subject to}$$

$$\sum_{i=1}^{N} p_i c_i = I.$$

- The FOCs of this problem imply that consumer's demand for each variety is given by:

$$c_i = \left[\frac{p_i}{\mathcal{P}} \right]^{-\sigma} \frac{I}{\mathcal{P}},$$

where $\mathcal{P}$ is the aggregate price index:

$$\mathcal{P} = \left[\sum_{j=1}^{N} p_j^{1-\sigma} \right]^{\frac{1}{1-\sigma}} .$$

Firm's Problem

- The firm that produces variety i, has productivity z_i and faces a marginal cost m_i for each unit it produces.
- Firm i knows the demand function of consumers. Then it chooses the price p_i that maximizes its profits.
- **The most important assumption**: Even though p_i is part of the aggregate price index $\mathcal{P}$, firm i considers itself atomistic, i.e., its pricing decision does not affect $\mathcal{P}$.
- The firm's problem is then:

$$\max_{p_i, c_i} \;\; z_i p_i c_i - m_i c_i \;\; \text{subject to}$$

$$c_i = \left[\frac{p_i}{\mathcal{P}}\right]^{-\sigma} \frac{I}{\mathcal{P}}.$$

Firm's Problem

- If we substitute the demand function into the objective function:

$$\max_{p_i,c_i} \quad z_i \left[\frac{p_i^{1-\sigma}}{\mathcal{P}^{-\sigma}}\right] \frac{I}{\mathcal{P}} - m_i \left[\frac{p_i}{\mathcal{P}}\right]^{-\sigma} \frac{I}{\mathcal{P}},$$

  which yields the FOC:

$$z_i(\sigma - 1)P_i^{-\sigma} = m_i \sigma P_i^{-\sigma-1}.$$

- Then, the optimal price firm i chooses for its variety is:

$$p_i^\star = \left[\frac{\sigma}{\sigma - 1}\right] \frac{m_i}{z_i}.$$

Monopolistic Competition and NKM

- The monopolistic competition framework allows us to understand something that we already saw in the data: why in a given period can prices of similar goods be different?
  - Because firms may differ in both their marginal costs and productivity!
  - Hence, a model that allows for **Idiosyncratic Shocks** (meaning, the shocks that firm i faces may differ from the one j receives) allows us to generate price heterogeneity.
  - We will mainly set the marginal cost to be equal across all firms, but will allow idiosyncratic productivity shocks that follow some distribution with mean μ and variance σ^2.

The Calvo Model

- The first NKM we will analyze is a version of the Calvo model.
  - Calvo-type models assume that each firm has to decide a price in period t and with probability θ (determined exougenously) it will be able to change its price again at $t+1$.
  - The interesting part of the model is that firms will receive a productivity shock in each period.
  - Hence, firms will most likely want to adjust their price, but a fraction $1-\theta$ of firms will not be able to do so.

The Calvo Model

- NKM are typically written in recursive form (since they are easy to handle this way).
- Which are the states that a firm considers?
  - Productivity shocks,
  - The price level $\mathcal{P}$.
  - The price the firm chose last period.
- Notice that there are two types of firms:
  - The ones that can change their price at the current period. Their value function is $V^C(\cdot)$.
  - The ones that cannot change their price. They have $V^F(\cdot)$ as value function.

The Calvo Model

- To simplify notation, we denote by $\pi(z_i, \mathcal{P}, p_i)$ the profits the firm can make in the current period whenever states are $(z_i, \mathcal{P}, p_i)$:

$$\pi(z_i, \mathcal{P}, p_i) = z_i \left[\frac{p_i^{1-\sigma}}{\mathcal{P}^{-\sigma}}\right] \frac{I}{\mathcal{P}} - m_i \left[\frac{p_i}{\mathcal{P}}\right]^{-\sigma} \frac{I}{\mathcal{P}}.$$

- Imagine we are considering a firm i that can choose the value of its price p_i (i.e. if it decides so, it can change its price). What problem does this firm faces?
  - The firm wants to choose p_i such that it maximizes current profits.
  - However, the firm must consider that in the future it will not be able to change p_i with probability θ.
  - Hence, the firm must incorporate this into its decision problem of p_i, discounting the future at some rate β.[2]

[2] You can think this discount factor as some type of interest rate used to discount future profits.

The Calvo Model: The Problem for $V^C(\cdot)$

- Then, a firm that can currently change its price, solves the following problem:

$$V^C(z_i, \mathcal{P}) =$$

$$\max_{p_i} \left\{ \pi(z_i, \mathcal{P}, p_i) + \beta \ \mathbb{E}_{z_i', \mathcal{P}'} \left[(1-\theta) V^F(z_i', \mathcal{P}', p_i) + \theta V^C(z_i', \mathcal{P}') \right] \right\} =$$

$$\max_{p_i} \left\{ \pi(z_i, \mathcal{P}, p_i) + \beta \ \sum_{z_i'} \sum_{\mathcal{P}'} f(z_i' | z_i) f(\mathcal{P}' | \mathcal{P}) \left[(1-\theta) V^F(z_i', \mathcal{P}', p_i) + \theta V^C(z_i', \mathcal{P}') \right] \right\}.$$

The Calvo Model: The Problem for $V^F(\cdot)$

- A firm that currently **cannot** change its price, faces the following problem:

$$V^F(z_i, \mathcal{P}, p_i^-) =$$

$$\pi(z_i, \mathcal{P}, p_i^-) + \beta \ \mathbb{E}_{z_i', \mathcal{P}'} \left[(1-\theta) V^F(z_i', \mathcal{P}', p_i^-) + \theta V^C(z_i', \mathcal{P}') \right].$$

- Notice that this firm cannot choose anything the current period!

Calvo Model: Intuition

- What is going on in this model?
- The firm has uncertainty about the future in two ways:
  - First, it does not know if it will be able to change its price.
  - Also, its productivity may shift, which may lead to higher or lower profits.
- Hence, what firms do in a Calvo setting is to choose very carefully p_i^1 since it must be such that, if they are unlucky and cannot change it in the future, it insures them if they receive a bad productivity shock.

Calvo Model: Simulation

- Let us assume that the income consumers have is $I = 10$, that productivity has an AR(1) distribution $log(z_i') = 0.6log(z_i) + \epsilon_z$, and that the price level follows $\mathcal{P}' = 0.96\mathcal{P} + \epsilon_{\mathcal{P}}$. Finally assume that $\theta = 0.15$.

- I will simulate this model for $N = 1,000$ firms and $T = 100$ periods to give further insight (and help fix ideas). I want to answer:

  1. What is the distribution of the magnitude of price changes? What about the frequency?
  2. Compare the co-movement of the average consumption, prices, and shocks through time.

Calvo Model: Magnitude of Price Changes

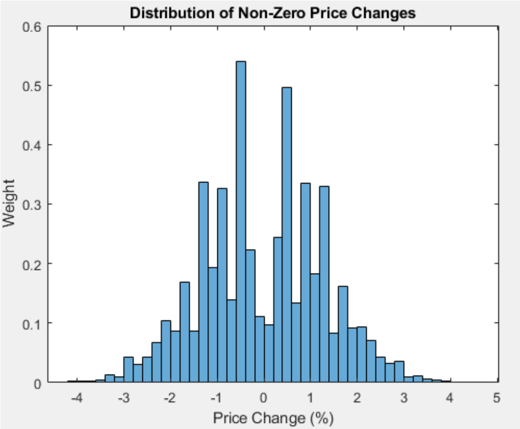

Figure: Magnitude of Price Changes in the Calvo Model

Calvo Model: Variables Through Time

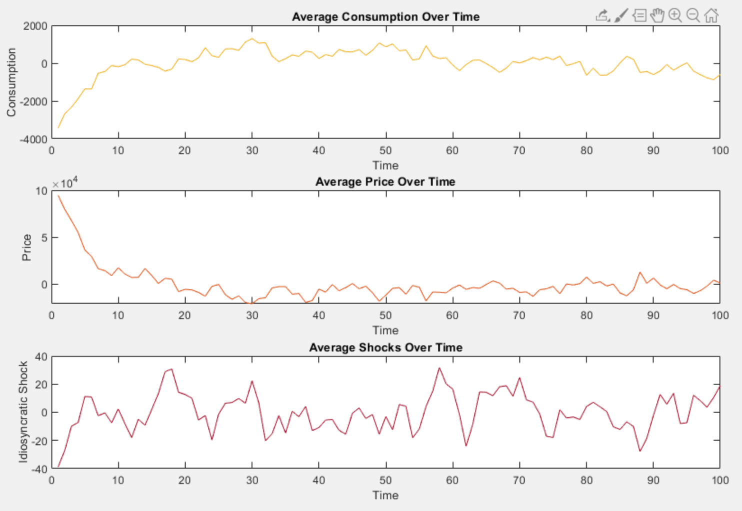

Figure: Consumption, Prices, and Shocks Evolution

Calvo Model: Welfare Analysis

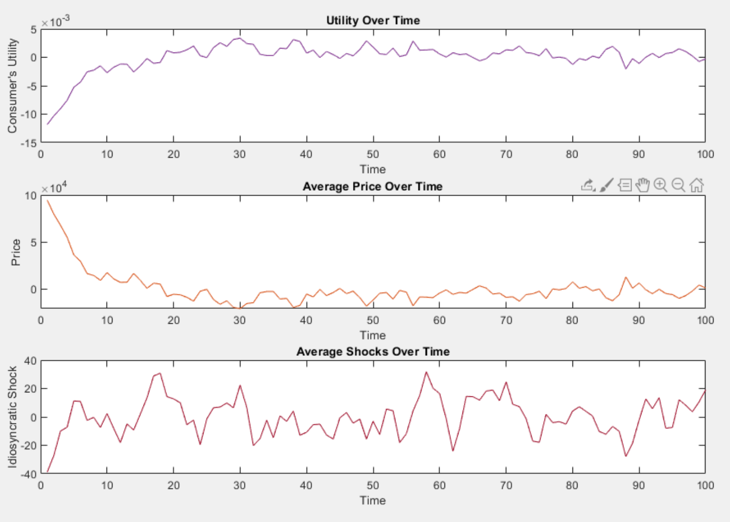

Figure: Welfare in the Calvo Model

Calvo Model: Simulation Results

- Here are some statistics that will allow us to understand how this model works.

Aggregate Prices Mean	0.0011
Aggregate Productivity Shocks Mean	5.4817
Aggregate Consumption Mean	0.0198
Aggregate Prices Variance	0.0000000030
Aggregate Productivity Shocks Variance	0.0056
Aggregate Consumption Variance	0.00005
Autocorrelation Aggregate Prices	0.9291
Autocorrelation Aggregate Productivity Shocks	0.6078
Autocorrelation Aggregate Consumption	0.9096
Correlation Utility and Aggregate Prices	-0.9960
Correlation Utility and Aggregate Shocks	0.2924

Table: Statistics of the Simulated Calvo Model

Calvo Model: The Role of θ

- What happens to the magnitude of price changes and the rest of the model's variables as θ approaches one?
- If you were a social planner and could choose θ, which one would you choose and why?
- Which public policies can influence θ?

Calvo Model: The Role of θ

- We now present a simulated model, considering the same parameters as before, but considering that every firm can change their price ($\theta = 1$).

Aggregate Prices Mean	0.0001
Aggregate Productivity Shocks Mean	5.4939
Aggregate Consumption Mean	0.0010
Aggregate Prices Variance	0.00000000010
Aggregate Productivity Shocks Variance	0.0018
Aggregate Consumption Variance	0.00000000010
Autocorrelation Aggregate Prices	0.7689
Autocorrelation Aggregate Productivity Shocks	0.8572
Autocorrelation Aggregate Consumption	0.6152
Correlation Utility and Aggregate Prices	-1.0000
Correlation Utility and Aggregate Shocks	-0.1032

Table: Statistics of the Simulated Calvo Model with $\theta = 1$

Calvo Model: The Role of θ

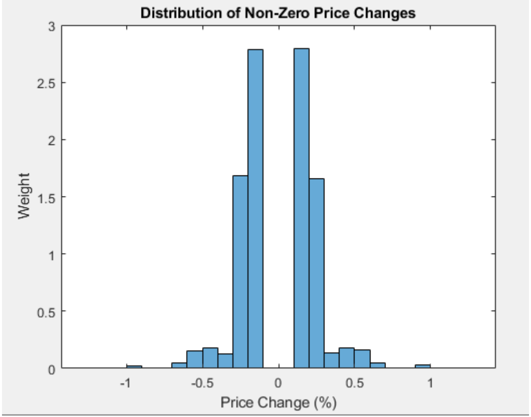

Figure: Magnitude of Price Changes in the Calvo Model with $\theta = 1$.

Calvo Model: Frequency of Price Changes

- What can we say about how often do price change (frequency of price changes) in the Calvo model?
  - Not much!
  - Remember, the parameter θ governs how often can firms change their price, and in this model it is **exogenous and fixed**.
  - If we want to endogeneize the frequency of adjustments, we need to use a **state-dependent model**.

Golosov-Lucas Menu Cost Model

- We now turn to analyze a **state-dependent model**: The Golosov-Lucas Menu Cost Model.
    - **Central Idea**: It is costly to change prices.
    - Think of Wal-Mart: if they want to adjust their prices every day they need to print again all their price tags.
    - Then, this model internalizes two problems that a firm faces:
        - How often it is optimal to change prices?
        - Whenver prices change, by how much do they adjust?

Menu Cost Models: Intuition

- Do firms need to change their price?
  - Consider a setting like the one in the Calvo model.
  - Firms face idiosyncratic shocks and respond to movements in the aggregate price.
  - If a firm never changes its price, it is internalizing the effects of productivity shocks it receives.
  - If the firm always changes its price, it is traspassing this shock to consumers.
  - If a price is fixed for a given period of time, then the firm considers an **Inaction Zone**:
    - While the shocks it receives aren't too big, the firm will internalize the shock cost.
    - Why? Because changing the price (re-printing its tags) is more costly than the effects of the shock.
    - The firm will not change its price **until** it receives a sufficiently large shock.

Menu Cost Models: Set-Up

- As in the Calvo model, each firm will face a demand given by:

$$c_i = \left[\frac{p_i}{\mathcal{P}}\right]^{-\sigma} \frac{I}{\mathcal{P}},$$

and will receive idiosyncratic productivity shocks z_i.

- **Menu Costs**: Let p_i^- be the price the firm had in the previous period and let p_i be its current pricing decision.
  - We assume that the firm incurrs in a cost χ whenever $p_i^- \neq p_i$.
  - You can thing χ as the cost of re-printing all the firm's price tags.

Menu Cost Models: Set-Up

- Let $V_i(\cdot)$ be the value function for firm i.
- Which are the states for a firm in this model?
  1. Productivity Shock z_i.
  2. Aggregate Prices $\mathcal{P}$.
  3. The price p_i^- the firm chose during the previous period.

Menu Cost Models: The Problem of a Firm

- Then, firm i solves the following problem:

$$V_i(z_i, \mathcal{P}, p_i^-) =$$

$$\max_{p_i} \left\{ \pi(z_i, \mathcal{P}, p_i) - \chi \ 1_{p_i \neq p_i^-} + \beta \ \ \mathbb{E}_{z_i', \mathcal{P}'} \left[V_i(z_i', \mathcal{P}', p_i) \right] \right\},$$

where $1_{p_i \neq p_i^-}$ is a function equal to one only if p_i (the price that the firm is currently choosing) is different from p_i^-.

Menu Cost Models: Simulation

- Now, I present a simulation of the menu cost model.
- Just as before, let us assume that the income consumers have is $I = 10$, that productivity has an AR(1) distribution $log(z_i') = 0.6 log(z_i) + \epsilon_z$, and that the price level follows $\mathcal{P}' = 0.96\mathcal{P} + \epsilon_{\mathcal{P}}$. Finally assume that $\chi = 0.10$.
- I will simulate this model for $N = 1,000$ firms and $T = 100$ periods to give further insight (and help fix ideas). I want to answer:
  1. What is the distribution of the magnitude of price changes? What about the frequency?
  2. Compare the co-movement of the average consumption, prices, and shocks through time.
  3. Is there a difference between a menu cost model and a Calvo model?

Menu Cost Model: Magnitude of Price Changes

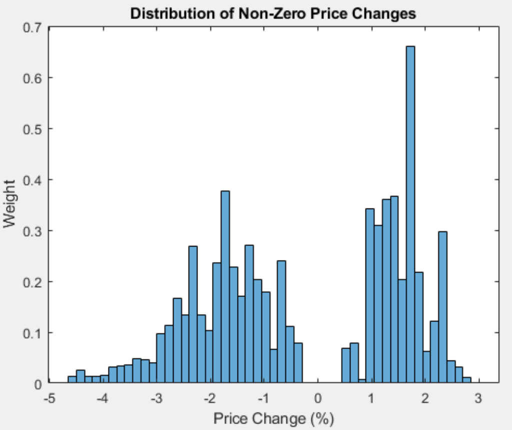

Figure: Magnitude of Price Changes in the Menu Cost Model

Menu Cost Model: Aggregates Through Time

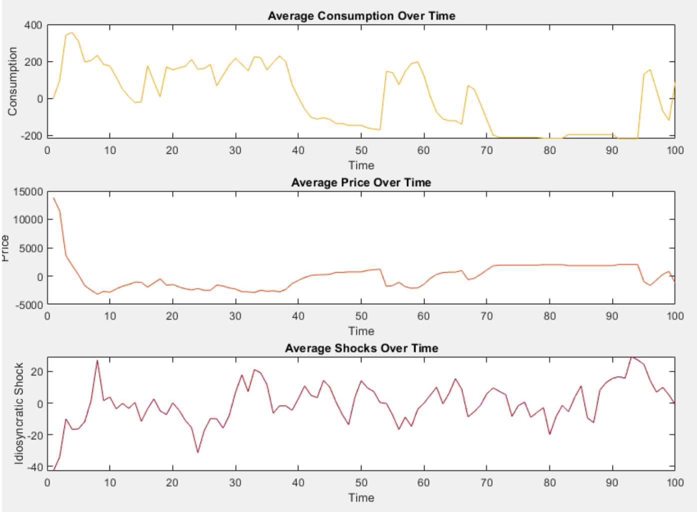

Figure: Evolution of Aggregates in the Menu Cost

Menu Cost Model: Welfare

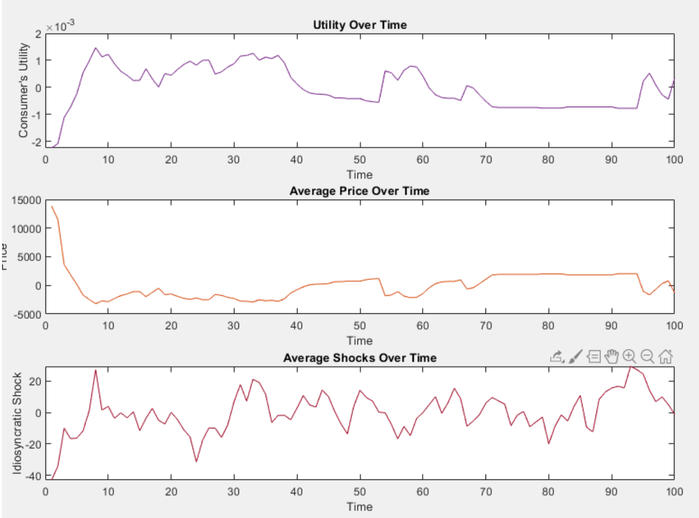

Figure: Evolution of Welfare in the Menu Cost

Menu Cost Model: Pricing of a Firm

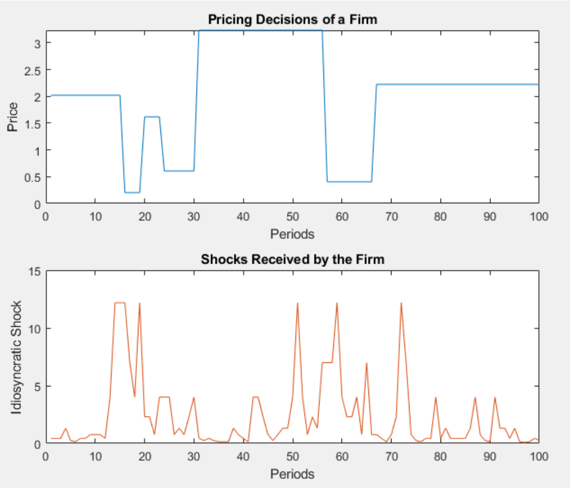

Figure: Pricing of a Firm.

Menu Cost Model: Pricing of a Firm

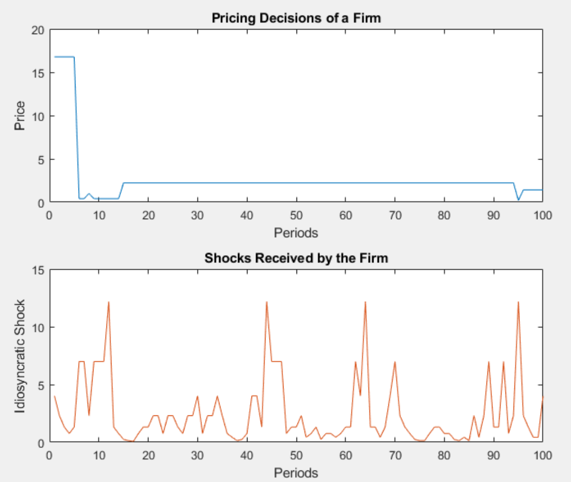

Figure: Pricing of a Firm.

Menu Cost Model: Pricing of a Firm

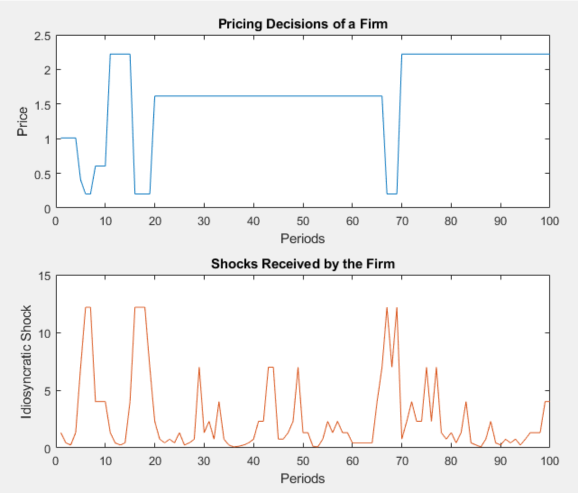

Figure: Pricing of a Firm.

Menu Cost Model: Simulation Results

- Here are some statistics that will allow us to understand how this model works.

Aggregate Prices Mean	0.0016
Aggregate Productivity Shocks Mean	5.5122
Aggregate Consumption Mean	0.0145
Aggregate Prices Variance	0.0000001564
Aggregate Productivity Shocks Variance	0.0063
Aggregate Consumption Variance	0.000002
Autocorrelation Aggregate Prices	0.9054
Autocorrelation Aggregate Productivity Shocks	0.7022
Autocorrelation Aggregate Consumption	0.8861
Correlation Utility and Aggregate Prices	-0.9211
Correlation Utility and Aggregate Shocks	0.0950
Average Duration of Prices	13.4001
Implied Frequency	0.0746

Table: Statistics of the Simulated Menu Cost Model

Menu Cost Models: The Role of χ

- What happens to duration as χ increases?

χ Parameter	Mean Duration of Prices
0.05	8.67
0.10	13.40
0.50	30.57
1	78.50
$\chi \to \infty$	∞
$\chi \to 0$	1

Table: Duration and χ

- What about its relationship with the magnitude of price adjustments?
- What is the optimal χ?

Towards a General Equilibrium Framework[3]

- The models we have seen so far, do not have all the components to be a complete general equilibrium framework.
- Why?
  - First, we are not explaining the nature of household's income (which in equilibrium should include profits, given that they are not zero, why?).
  - Also, we are still quite not explaining how firms produce (capital, labor decisions which usually are part of GE frameworks).
- Most important missing ingredient: we are assuming a given process for $\mathcal{P}$ which may or not be **consistent** with the actual movement of prices.
  - We need to impose **Rational Expectations** (or other type of consistent expectations).

[3] Although I making this point until this slide, let me be clear that all the models I previously simulated consider rational expectations.

Rational Expectations

- NKM as well as other frameworks heavily rely on the following: firms need to be able to predict $\mathcal{P}$ either by knowing exactly how it will evolve or by forming some beliefs (expectations) about its evolution.
- Why predicting $\mathcal{P}$ is important? Because in this model it affects each firm's demand.
- We say that firms (or in general agents) have **Rational Expectations** whenever their beliefs about a future (usually aggregate) variable, in this case $\mathcal{P}$, exactly coincides with the process that determines it.
- Even though this might sound reasonable, assuming rational expectations poses a big computational problem to us as modelers: we need to be able to come up with a process for $\mathcal{P}$ that, when computing the model and each firms' p_i, it is the case that not only $\mathcal{P}$ is the result of aggregating each p_i, but also how p_i' behaves matches with $\mathcal{P}'$!!

Rational Expectations

- There are several papers that deal with different approaches on this issue, and mainly what NKM modelers do is to do an iterative process:
  - They assume that $\mathcal{P}$ evolves according to, lets say $f_1(\mathcal{P})$.
  - Given this, they compute the model and see how p_i and its aggregate evolves.
  - If it coincides with f_1, then we stop and have a Rational Expectations model.
  - If it does not coincide, then we take the implied evolution of $\mathcal{P}$ in the model and name it f_2.
  - We compute the model again, and compare.
  - We continue to do this until the implied f of the model, coincides with the assumed evolution for $\mathcal{P}$.

Rational Expectations

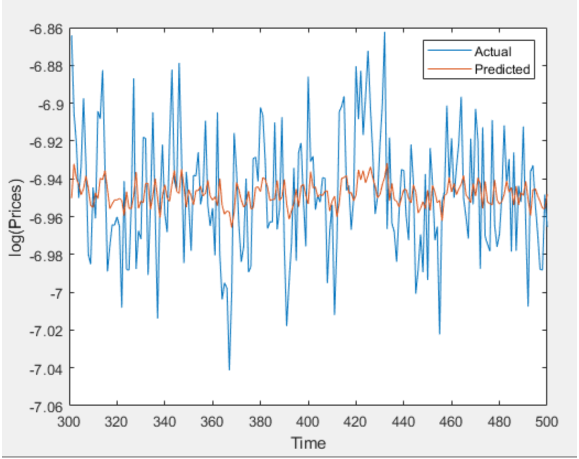

Figure: Initial Expectations vs Actual Aggregate Prices.

Rational Expectations

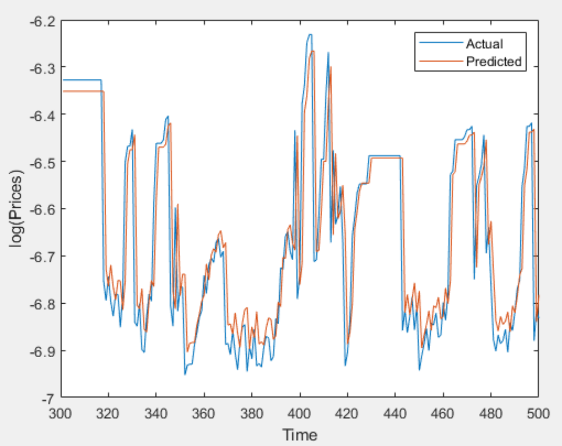

Figure: Rational Expectations and Actual Aggregate Prices.

Rational Expectations

- Why do we care that much about how firms (agents) predict aggregates?
- Because they are crucial to determine policy approaches!
  - We need to be able to say how will agents optimally behave whenever we introduce a policy, and in order to say this, we need to understand how they form their expectations.
- We will not go into further details, just keep in mind that every model that considers the interaction between idiosyncratic and aggregate variables, will have this "estimation" of expectations problem.

Thinking Critically About The Model

- What are Neo-Keynesian models useful for?
- What are the main assumptions of Neo-Keynesian models?
  - Do we believe these assumptions are realistic?
  - Which one would you criticize the most?
- What are the limitations of Neo-Keynesian models?
  - Give a concrete example of a situation where using a Neo-Keynesian model would not be useful.

Outline

Motivation

- Up to this moment, we have considered two models in which there is a mass of identical consumers (same preferences and endowments). We focused our attention on an infinitely lived representative agent.
  - Both models are useful to explain some features of the data.
  - But in the context of these models can we analyze central policy topics like inequality, life cycle, inefficiencies?
  - NO! Why?
- The Overlapping Generations Model (OLG) is a mayor workhorse in the macro literature that will provide a natural context to analyze issues that the classical model cannot explain.
  - Pioneered by Allais (1947), Samuelson (1958), and Diamond (1965).
  - Appealing since it allows us to integrate micro and macro data.
  - We will be able to analyze topics like: social security, source of market inefficiencies, the value of money, distributive effect of taxes, life cycle savings, among others.

Basic Ingredients of an OLG

- Time is discrete $t = 1, 2, ...$
- We will consider a single nonstorable consumption good.
- **Key feature**: individuals do not live forever.
  - For the moment, we will assume that individuals only live for two periods and then die.
  - Every period a new generation is born, with mass 1.
  - We will index generations by the period they are born: g^t will be the generation that was born at period t.
  - Each generation will be endowed every it lives period with units of consumption. Let (e_t^t, e_{t+1}^t) be the endowment that generation t receives at period t and at $t+1$.

Consumption Timing

- We denote (c_t^t, c_{t+1}^t) the consumption of generation t at periods t and $t+1$.
- Notice that every period t there are two generations alive:
  - **Young Generation**: born at period t, endowed with e_t^t who consume c_t^t.
  - **Old Generation**: born at period $t-1$, endowed with e_t^{t-1} who consume c_t^{t-1}.
- There is an **Initial Old Generation** at period $t=1$ who is endowed e_1^0 and consumes c_1^0.

Consumption Timing

generation\time	1	2	...	t	$t+1$
0	(c_1^0, e_1^0)				
1	(c_1^1, e_1^1)	(c_2^1, e_2^1)			
$\vdots$			$\ddots$		
$t-1$				(c_t^{t-1}, e_t^{t-1})	
t				(c_t^t, e_t^t)	(c_{t+1}^t, e_{t+1}^t)
$t+1$					$(c_{t+1}^{t+1}, e_{t+1}^{t+1})$

Figure: Timing in an OLG Model

- Alternative and interesting interpretation: an OLG model can be seen as a framework where there are infinitely lived consumers that **only care about consumption in two periods**.
  - By reading the model in this way, we can see that in an OLG context there are **Incomplete Markets**!
  - This suggests that Pareto Optimality may not hold in OLG models.
  - We will see that, in many cases, equilibria are inefficient.

Preferences and Feasibility

- Preferences of generation t are given by:

$$U_t(c) = log(c_t^t) + \beta log(c_{t+1}^t).$$

- The initial old have preferences $U_0(c) = log(c_1^0)$.
- We say a consumption allocation $c_1^0, \{(c_t^t, c_{t+1}^t)\}_{t=1}^{\infty}$ is **feasible** if:

$$c_t^{t-1} + c_t^t = e_t^{t-1} + e_t^t \quad \text{for every} \quad t \geq 1.$$

Selfishness

- What are the incentives for each generation?
  - The young generation would like to consume (obviously), however, it knows that they will live for one more period. Hence, young people have incentives to save for the future.
  - The old generation also values consumption and they die the next period, hence they **do not have incentives to do anything but consume**.
  - Up until this point, even though the young generation will like to save for the future they cannot do it since there is no one to trade with!
  - If we leave the model like this it will always imply an **autarkic equilibrium**, where generations only consume their endowments and there is no trade.
- If we want to induce trade between generations (which is a feature we would like since trade usually increases welfare) we need to introduce a mechanism that promotes it.
  - Solution: **Money**!!

Money and OLG

- Why money promotes trade?
  - Simple: it is a **promise of future payment**.
  - If the initial old have money to trade, they can give it to the initial young at $t = 1$, which they when old ($t = 2$) can trade with the young in exchange for consumption goods.
  - Hence, money is a certificate that guarantees you as a young person that, when old, you will receive consumption goods in exchange for it.
  - Money is memory (Kocherlakota, 1998).
- In this model, money is a **bubble**: even though it has no intrinsic value (it is literally a piece of paper), people use it to trade and the market valuates it at a positive price.
  - This can only occur in the context of incomplete markets. There is a famous theorem by Samuelson which states that in an economy with infinitely lived agents and no frictions money is never valued in equilibrium.
- What is the role of monetary policy then?

Market Structure

- We will mainly focus on markets that allow sequential trading. Hence, there are markets for the consumption good in every period.
- In addition, there is an asset through which generations can do their savings.
- We will denote a_t^t the savings done by generation t which will be delivered/payed in period $t+1$.
- Let r_{t+1} be the interest rate related to such assets between period t and $t+1$.

Sequential Markets Equilibrium

- Let $m \geq 0$ be the amount of money available in the economy.
- A sequential markets equilibrium is an allocation $\hat{c}_1^0, \{(\hat{c}_t^t, \hat{c}_{t+1}^t, \hat{a}_t^t)\}_{t=1}^{\infty}$ and a sequence of interest rates $\{\hat{r}_t\}_{t=1}^{\infty}$ such that:

1. Given interest rates, $(\hat{c}_t^t, \hat{c}_{t+1}^t, \hat{a}_t^t)$ solves generation's t problem:

$$\max_{\{c_t^t, c_{t+1}^t, a_t^t\}} \; log(c_t^t) + \beta log(c_{t+1}^t) \; \text{ subject to}$$

$$c_t^t + a_t^t \leq e_t^t,$$

$$c_{t+1}^t \leq e_{t+1}^t + (1 + \hat{r}_{t+1})a_t^t.$$

2. Given interest rates, $\hat{c}_1^0$ solves the initial old problem:

$$\max_{\{c_1^0\}} \; log(c_1^0) \; \text{ subject to}$$

$$c_1^0 \leq e_1^0 + (1 + \hat{r}_1)m.$$

3. For all $t \geq 1$ markets clear:

$$\hat{c}_t^{t-1} + \hat{c}_t^t = e_t^{t-1} + e_t^t.$$

What are Savings in this Economy?

- At every period:

$$c_t^t + a_t^t = e_t^t \quad c_t^{t-1} = e_t^{t-1} + (1 + r_t)a_{t-1}^{t-1}.$$

- If we add these budget constraints:

$$c_t^{t-1} + c_t^t + a_t^t = e_t^{t-1} + e_t^t + (1 + r_t)a_{t-1}^{t-1},$$

which then by market clearing becomes:

$$a_t^t = (1 + r_t)a_{t-1}^{t-1}.$$

- In particular, at $t = 1$:

$$a_1^1 = (1 + r_1)m,$$

which implies that:

$$a_t^t = \prod_{\tau=1}^{t}(1 + r_\tau)m.$$

- Then r measures the **return of money**.

Computing Arrow-Debreu Prices

- Remember that in an Arrow-Debreu market structure, p_t is the price of consumption in period t in terms of the initial period goods.
- How can we recover these prices?
  - Easy! As we have been doing all this time:

$$\frac{p_t}{p_{t+1}} = 1 + r_{t+1} \quad \frac{1}{p_1} = 1 + r_1$$

- Once we have these prices, we know that the consumption allocation is the same in either market structure.
- We can give an additional interpretation to this relationship between p and r in this model:
  - Remember r_{t+1} is the return on money.
  - In this case:

$$1 + r_{t+1} = \frac{p_t}{p_{t+1}} = \frac{1}{1 + \pi_{t+1}}.$$

  - Then (using a log approximation) $r_{t+1} = -\pi_{t+1}$.

Offer Curves

- How do we actually find an equilibrium for this economy?
- Gale (1973) proposed a nice algorithm to do this.
- We will assume that the endowment for the young generations is always e_1 while for the old it is always e_2.
- Let $c_t^t(p_t, p_{t+1}), c_{t+1}^t(p_t, p_{t+1})$ denote the optimal consumption of generation t at periods t and $t+1$ as a function of prices (like in Eco III).
- Let us consider the (Walrasian) excess demand functions:

$$y^t(p_t, p_{t+1}) = y^t\left(\frac{p_{t+1}}{p_t}\right) = c_t^t(p_t, p_{t+1}) - e_1$$

$$z^t(p_t, p_{t+1}) = z^t\left(\frac{p_{t+1}}{p_t}\right) = c_{t+1}^t(p_t, p_{t+1}) - e_2$$

Offer Curves

- As we vary p_{t+1}/p_t we obtain what Gale called **Offer Curve**.

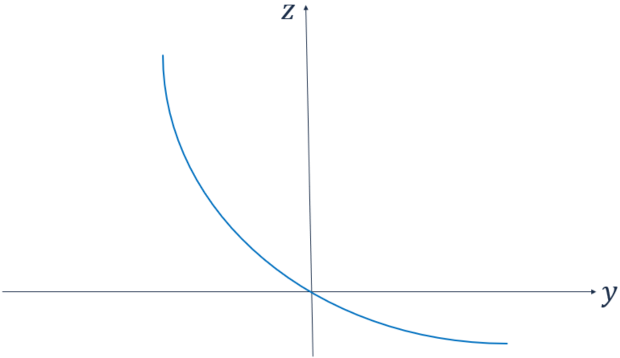

Figure: Offer Curve

Offer Curves

- Since this excess demand functions have to satisfy budget constraint:

$$p_t y^t(p_t, p_{t+1}) + p_{t+1} z^t(p_t, p_{t+1}) = 0,$$

which implies:

$$\frac{z^t(p_t, p_{t+1})}{y^t(p_t, p_{t+1})} = -\frac{p_t}{p_{t+1}}$$

- For the initial old:

$$z^0(p_1) = \frac{m}{p_1}.$$

- Then, given a choice of p_1 we know z^0 and can iterate to get all the future sequence of prices!

Offer Curves: Computing Equilibrium Prices

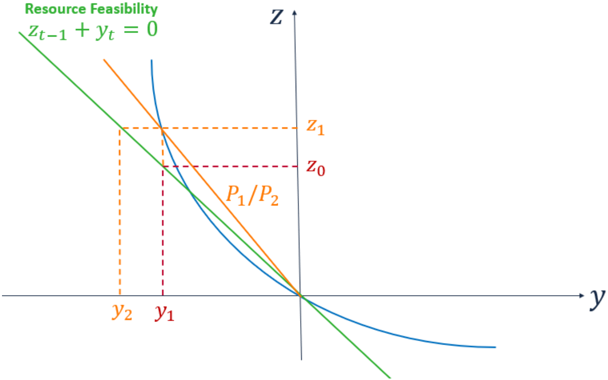

Figure: Offer Curve

Offer Curves: Some Important Remarks

- Notice that if we change p_1 the entire sequence of prices and allocations is shifted!
- Then, we can see that in an OLG model there is a multiplicity of equilibria.
  - This is not a desirable property of a model.
  - Which of the (possibly infinite number) of equilibria is "the best"?
  - Are all equilibria efficient? Seems unlikely.
- What is the role of monetary policy?

The Balasko-Shell Theorem

- When is an equilibrium efficient?
- Let $\{\hat{p}_t\}_{t=1}^{\infty}$ denote the sequence of equilibrium prices.
  - The **Balasko-Shell Theorem** states that $\{\hat{p}_t\}_{t=1}^{\infty}$ induces an equilibrium allocation that is Pareto Efficient if and only if

$$\sum_{t=1}^{\infty} p_t < \infty.$$

- Hence, if prices "explode", $\sum_{t=1}^{\infty} p_t = \infty$, the implied equilibrium is not efficient.
  - Remember that prices and interest rates (return of money) are inversely related in equilibrium.
  - Hence, prices exploding in equilibrium is equivalent to **interest rates being too low (or even negative)**.
  - **Key Intuition**: If interest rates are too low, trade is desincentivized and hence, the economy is "close" to the autarkic equilibrium, that we know is inefficient (in general).

Numerical Example

- Imagine an OLG economy where $e_t^t = 30$ and $e_{t+1}^t = 4$ for every t. Preferences are given by $log(c_t^t) + log(c_{t+1}^t)$.

1. Set up the problem that the consumer solves and find the consumption demands.
2. Suppose there is no money in this economy. What are the equilibrium prices and allocations?
3. Is this equilibrium efficient?
4. Find the Offer Curve equation for this economy.
5. Suppose $m = 10$ and $p_1 = 1$. Find the sequence of equilibrium prices.
6. Is this equilibrium Pareto Optimal?

Numerical Example: Question 1

- Generation t solves:

$$\max_{c_t^t, c_{t+1}^t} \; log(c_t^t) + log(c_{t+1}^t) \; \text{ subject to}$$

$$p_t c_t^t + p_{t+1} c_{t+1}^t = 30 p_t + 4 p_{t+1}.$$

- Notice this is a standard Eco III problem where the consumer demands two goods and has an income (with Cobb-Douglas utility). Hence, demands are given by:

$$c_t^t(p_t, p_{t+1}) = \frac{30p_t + 4p_{t+1}}{2p_t} \quad c_{t+1}^t(p_t, p_{t+1}) = \frac{30p_t + 4p_{t+1}}{2p_{t+1}}$$

Numerical Example: Question 2

- Remember, if $m = 0$ then there is no trade in this economy (why?). Then, in equilibrium each generation will consume its endowment. Hence:

$$c_t^t = 30 \quad c_{t+1}^t = 4.$$

- Using any of the demand functions, we can attain the equilibrium prices:

$$30 = c_t^t = 15 + \frac{2p_{t+1}}{p_t},$$

implying that:

$$\frac{p_{t+1}}{p_t} = \frac{15}{2},$$

and hence:

$$p_t = \left[\frac{15}{2}\right]^t.$$

Numerical Example: Question 3

- To assert if this equilibrium is or not Pareto efficient, we need to compute:

$$\sum_{t=1}^{\infty} p_t = \sum_{t=1}^{\infty} \left[\frac{15}{2}\right]^t = \infty,$$

since $\frac{15}{2} > 1$.

- Hence, by the Balasko-Shell Theorem, this equilibrium is not efficient.
- Intuition?

Numerical Example: Question 4

- Let $P = p_t / p_{t+1}$. Then the excess demand functions are given by (we drop the t indicators):

$$y = \frac{2}{P} - 15,$$

$$z = 15P - 2.$$

- Hence $P = \frac{2}{y+15}$ and therefore:

$$z = 15\left(\frac{2}{y+15}\right) - 2 = \frac{30 - 2(y+15)}{y+15} = -\frac{2y}{y+15}.$$

- Then the offer curve of this economy is:

$$z = -\frac{2y}{y+15}.$$

Numerical Example: Question 5

- Now, since $m = 10, p_1 = 1$ then $z^0 = 10$. By market clearing conditions this implies that $y^1 = -10$.
- Then, by the offer curve and the price equation:

$$z^1 = \frac{2(10)}{-10+15} = 4 \quad P_1 = \frac{p_1}{p_2} = \frac{1}{p_2} = \frac{-z^1}{y^1} = \frac{2}{5}.$$

- Now, since $z^1 = 4$ this implies that $y^2 = -4$ and hence:

$$z^2 = \frac{2(4)}{-4+15} = \frac{8}{11} \quad P_2 = \frac{p_2}{p_3} = \frac{-z^2}{y^2} = \frac{2}{11}.$$

- Since $z^2 = 8/11$ then $y^3 = -8/11$ and therefore:

$$z^3 = \frac{16}{157} \quad P_3 = \frac{p_3}{p_4} = \frac{22}{157}....$$

Discussing the OLG Model Assumptions

- As we have seen, the particular assumptions made for the OLG framework have some interesting consequences.
- How important are these assumptions in order for the results to still hold?
- In particular, we will discuss the role of:
  1. Households being selfish vs being altruistic.
  2. Agents living a finite number of periods vs living forever.
  3. Agents knowing in which period they die vs not knowing this.

Altruism

- One could criticize the OLG model since many of its interesting results rely on the "selfishness" feature of the model.
- What happens if we allow room for altruism?
  - In particular, we could think that generation $t-1$ are the parents of generation t, and hence generation $t-1$ cares on the well being of its children.
  - We introduce the notion of **bequest**: units of consumption good a parent (old generation $t-1$) can give its children (young generation t) after they die.
  - We assume the bequest is non-strategic.
  - Let $b_t^{t-1} \geq 0$ be the bequest generation $t-1$ leaves to generation t. Then, in period $t+1$ generation t (which is now old) will have as income:

$$e_{t+1}^t(1+r_{t+1})a_{t+1}^t + b_t^{t-1}.$$

Altruism

- Then, parents solve the following problem (notice that their parents, generation $t-2$, left them a bequest of size b_t^{t-1} which they take as **given**):

$$V_{t-1}(b_t^{t-1}) = \max_{\{c_{t-1}^{t-1}, c_t^{t-1}, b_{t+1}^t, a_t^{t-1}\}} \left\{ log(c_{t-1}^{t-1}) + \beta log(c_t^{t-1}) + \alpha V_t(b_{t+1}^t) \right\}$$

subject to

$$c_{t-1}^{t-1} + a_t^{t-1} = e_{t-1}^{t-1},$$

$$c_t^{t-1} + b_{t+1}^t = e_t^{t-1} + (1 + r_t)a_t^{t-1} + b_t^{t-1}$$

- Where we can interpret $0 < \alpha < 1$ as the weight parents give in their own utility to the well-being of their children.

Altruism

- But then, starting from the initial old, notice the following (I will not write the budget restrictions, but remember they are there):

$$V_0(b_1) = \max\left\{log(c_1^0) + \alpha V_1(b_2^1)\right\}$$

$$= \max\left\{log(c_1^0) + \alpha \max\left\{log(c_1^1) + \beta log(c_2^1) + \alpha V_2(b_3^2)\right\}\right\}$$

$$= \max\left\{log(c_1^0) + \alpha\left[log(c_1^1) + \beta log(c_2^1)\right] + \alpha^2 V_2(b_3^2)\right\}$$

$$= \max\left\{log(c_1^0) + \alpha\left[log(c_1^1) + \beta log(c_2^1)\right] + \alpha^2 \max\left\{log(c_2^2) + \beta log(c_3^2) + \alpha V_3(b_4^3)\right\}\right\}$$

$$= \max\left\{log(c_1^0) + \alpha\left[log(c_1^1) + \beta log(c_2^1)\right] + \alpha^2\left[log(c_2^2) + \beta log(c_3^2)\right] + \alpha^3 V_4(b_5^4)\right\} \ldots$$

$$= \max\left\{\sum_{t=0}^{\infty} \alpha^t \tilde{u}(c_t)\right\}$$

- Hence, allowing altruism when parents care $\alpha > 0$ about their children's utility leads to the same problem that an infinitely lived representative agent solves! WOW!

Altruism: Final Remarks

- Then, assuming altruism leads households to behave **AS IF** they were infinitely lived, even though they actually live for two periods.

- This result is another justification commonly used in the literature for the infinitely lived representative agent assumption.

- Shall we use selfish or altruistic households?
    - **It depends on what you wish to highlight**!!
    - Assuming altruism is equivalent of assuming a representative agent, hence you recover Pareto Optimality of equilibria, complete markets, etc...
    - However, assuming altruistic households takes you away from an OLG framework. Then you can no longer rationalize inequality, the value of money, monetary policy, incomplete markets, etc...

- Important lesson: one has to seriously think about the assumptions one is making, since as you can now understand, they have important consequences!

Finite vs Infinite Life

- Another key assumption in the OLG framework is that **agents do not live forever**.

- Assuming this creates an heterogeneity within each period, since there are agents who are at different moments of their life cycle, and therefore have different incentives.
  - For example, in a world in which agents live for two periods, young agents have an incentive to save while old agents have an incentive to consume.
  - This creates a tension within the model, leading to incomplete markets, and therefore makes interesting to introduce mechanisms such as money.

- On the other hand, if agents lived forever, even in a context in which we have heterogeneous agents (e.g. a continuum of agents each with different endowments), they would have incentives to trade either with their "future selves" or with other agents.
  - This takes us back to the world of complete markets.

Exogenous Probability of Death

- What about we assumed that each agent in the economy has an exogenous probability of dying, given by $\lambda > 0$?
- Now, from the point of view of time t, the expected utility of consuming in $t+1$ is given by (we are assuming that if an agent dies she receives a utility of zero):

$$(1-\lambda)u(c_{t+1}) + \lambda 0.$$

- Hence, assuming the agent discounts the future at a rate β, the recursive problem the agent solves is:

$$V(a) = \max\{u(c) + \beta(1-\lambda)V(a')\}$$

$$c + a' = e + (1+r)a.$$

- Notice that this is the same problem that in the classical model, just with a different discount factor!

Production and OLG

- To close up our discussion on OLG models, we now analyze a production economy with OLG.
- Let N_t^t be the number of individuals of generation t that live at period t, and N_t^{t-1} the number of individuals of generation $t-1$ that live during t.
  - People do not die early: $N_t^t = N_{t+1}^t$.
- We assume the same type of preferences as before (with no altruistic component).
- Now, each individual of the young generation is endowed with one unit of labor. And we will assume (for simplicity) that they supply it inelastically to the market, receiving a wage in return.
  - The old generation does not work (they are retired).
- There is a representative firm with technology $F(K_t, N_t) = AK_t^{\alpha} N_t^{1-\alpha}$.

Production and OLG: Timing

- Within a period, the following occurs:

1. First, production takes place with the capital K_t that the now old generation $t-1$ saved during $t-1$. Production also requires labor which is provided by the young generation t for which they receive a wage w_t.
2. Once they are paid, the young generation decides how much to consume and how much to save. Savings are in the form of physical capital.
3. When period $t+1$ arrives, generation t is now old and does not work anymore. Instead, they receive as income an effective interest $1+r_{t+1}-\delta$ for each unit of capital they saved when young.
4. With this income, the now old generation t consumes. And at the end of the period this generation dies.

Generation's t Problem

- The problem that each person belonging to generation t solves is then:

$$\max \quad u(c_t^t) + \beta u(c_{t+1}^t) \quad \text{subject to}$$

$$c_t^t + a_{t+1}^t = w_t$$

$$c_{t+1}^t = (1 + r_{t+1} - \delta) a_{t+1}^t$$

Equilibrium

- In equilibrium, the following must be true for markets to clear:

$$N_t^t c_t^t + N_t^{t-1} c_t^{t-1} + K_{t+1} = AK_t^{\alpha}(N_t^t)^{1-\alpha} + (1-\delta)K_t,$$

$$K_{t+1} = N_t^t a_{t+1}^t + (1-\delta)K_t.$$

OLG and Production: Numerical Example

- Consider an Economy where each generation lives for three periods: they are young, middle-aged, and old. Each generation has a utility function:

$$log(c_t^t) + log(c_{t+1}^t) + log(c_{t+2}^t)$$

- Each generation is endowed with units of labor (l^y, l^m, l^o) which they supply inelastically to the labor market.

1. What is the problem a generation t solves?
2. How is the life cycle profile of consumption, income, and savings when $(l^y, l^m, l^o) = (5, 10, 0)$?
3. How is labor income and wealth inequality over time? Is there a difference?
4. How about when $(l^y, l^m, l^o) = (10, 10, 10)$?
5. Discuss the results of each model.

OLG and Production: Numerical Example

- Each generation solves:

$$\max \quad log(c_t^t) + log(c_{t+1}^t) + log(c_{t+2}^t) \quad \text{subject to}$$

$$c_t^t + a_{t+1}^t = w_t l^y,$$

$$c_{t+1}^t + a_{t+2}^t = w_{t+1} l^m + (1 + r_{t+1} - \delta) a_{t+1}^t,$$

$$c_{t+2}^t = w_{t+2} l^o + (1 + r_{t+2} - \delta) a_{t+2}^t,$$

$$c_t^t, \quad c_{t+1}^t, \quad c_{t+2}^t \geq 0.$$

- To solve this model, we need to also write the initial old and initial middle's problem as well as the market clearing conditions (please write them) and then use MATLAB.

- I now present the main results of solving the model in the computer.

Convergence Towards a Steady-State Equilibrium

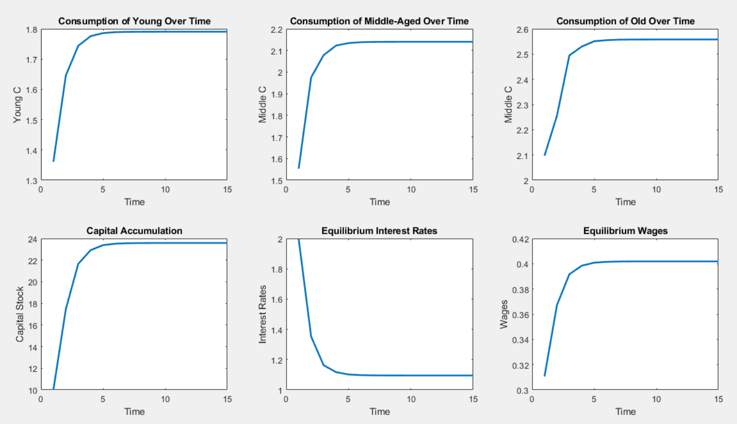

Figure: Aggregates Over Time

Life-Cycle Behavior

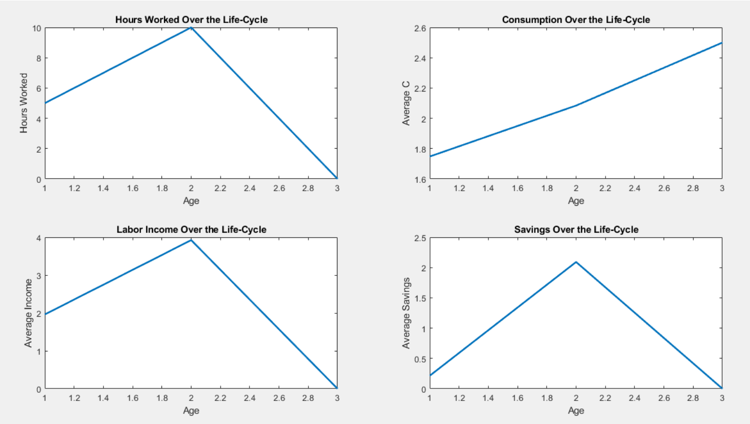

Figure: Life-Cycle Behavior (Averages)

Income and Inequality

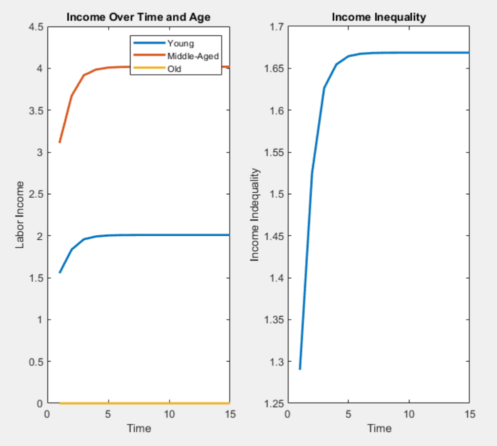

Figure: Income and Inequality over Time.

Wealth and Inequality

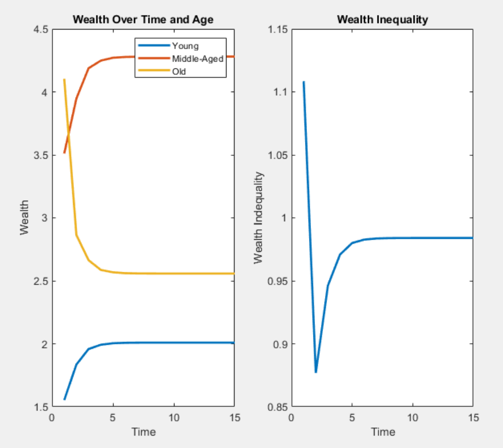

Figure: Wealth and Inequality over Time.

Life-Cycle Behavior With Different Labor Endowment

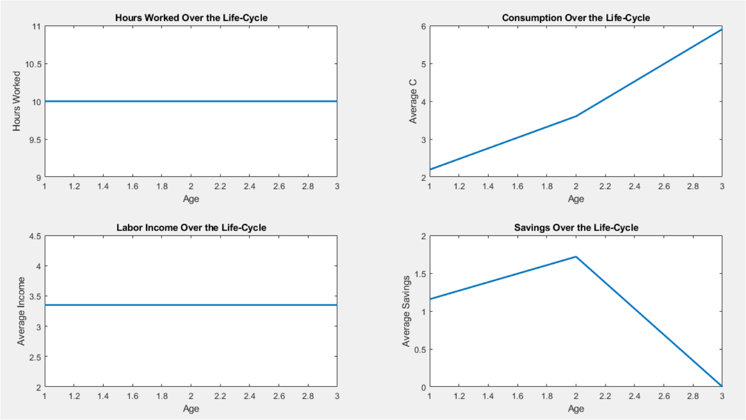

Figure: Life-Cycle Behavior Whenever Everyone Works the Same Time.

Wealth and Inequality

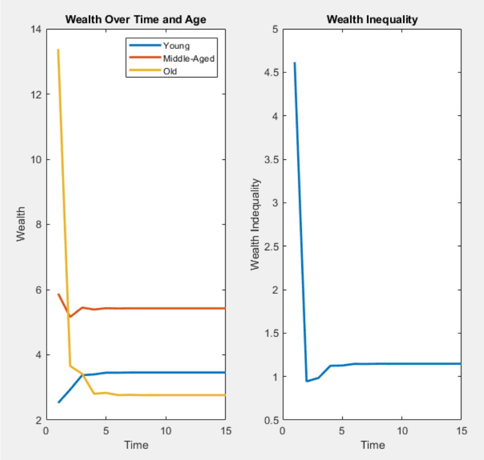

Figure: Wealth and Inequality over Time.

Final Remarks on OLG and Production

- Although we will not get into the (nasty) details, the following characteristics are true for an OLG production equilibrium:
    - Welfare theorems break down in many cases (as in the endowment case).
    - Fortunately, the Balasko-Shell theorem still holds.
    - There can be multiple equilibria, making this a complicated problem to solve (even in the computer).
    - This model is very attractive since its main implication is what is referred to as **Dynamic Inefficiency**: since interest rates are not what they are supposed to do (why?) generations either save more (or less) of what it would be "optimal".
    - There are several papers showing that these type of dynamic inefficiencies are present in some countries.
    - Hence, this suggest the need for policies that (for example through a tax) increase/reduce savings to achieve optimality.

Thinking Critically About The Model

- What is the OLG model useful for?
- What are the main assumptions of the OLG model?
  - Do we believe these assumptions are realistic?
  - Which one would you criticize the most?
- What are the limitations of the OLG model?
  - Give a concrete example of a situation where using the OLG model would not be useful.

Outline

Motivation

- So far we have studied three competitive equilibrium models in which agents supply labor to the market.
- Can any of these frameworks account for the unemployment we observe in the data?
  - No! Since they are models that, by definition, assume all markets clear.
  - In particular, the labor market clears: the amount of labor households offer is exactly the amount firms demand.
- We would like to answer the following questions about employment and unemployment:
  - Why is there unemployment?
  - Is unemployment inefficient?
  - What determines the distribution of wages?
  - Why does unemployment fluctuate over the business cycle?
  - Does the job market need regulation?

Endogenous Search Models

- We will focus on **equilibrium models of unemployment**.
  - In these kind of frameworks, unemployment does not imply that the labor market is not in equilibrium.
  - Unemployment will be an equilibrium **outcome**.
- How can we achieve unemployment in equilibrium?
  - We need to establish some type of friction in the labor market, i.e., we need to make costly either for the worker or the employee to use/hire labor.
  - We will introduce two frictions:
    1. Agents have to **search** for an employer/worker. Search will be costly to both.
    2. Once a worker and an employer find each other they have to **match**. In this case, they have to agree on the wage the worker will receive.

Bargaining

- In order to fully understand the model we are going to study, we will break it into pieces. First, we will analyze the problem of **Bargaining**.
- Once an employer and a worker found each other (we will study later how they find each other) they need to negotiate the wage the employer will pay the worker.
  - Each part involved in this bargaining has different incentives.
  - The worker has incentives to ask for a high wage, while for the employer, the lower is the wage she pays the better.
  - Both have an **Outside Option**: either can walk out of the negotiation and break the match.
  - What is the outside option for each part?
    - Employer: the threat that he can find another person willing to work at a lower wage.
    - Worker: the possibility to remain unemployed and receive an unemployment insurance.

Bargaining

- In the literature, there are two wide class of Bargaining models that are studied:
  - **Strategic Bargaining**: Developed in the context of game theory, where each agent has some beliefs about the other's position and post wages sequentially until the other either accepts, walks out, or proposes a new wage.
  - **Axiomatic Bargaining**: Originally developed by Nash (1950) in which the outcome of the negotiation is the solution to a maximization problem that considers both the benefits and outside options of both parts.
- We will focus on the second type of bargaining structure, which is also usually referred to as **Nash Bargaining**.

Nash Bargaining

- Imagine there are two agents, a buyer (B) and a seller (S), who are going to bargain over an object the seller owns and the buyer wants to get.
- The seller has a valuation $v_s \geq 0$ of the object while the buyer valuates it at $v_b \geq 0$.
  - S has the option to not sell, and threatens B to find another buyer that will pay v_s.
  - B has the option to walk out, threatening with the possibility to find another seller which gives him the object valuated at v_b.
- Suppose they are bargaining over the price p that the buyer must pay to the seller for him to get the object.
- Each one has an (indirect) utility function that depends on p $U_s(\cdot), U_b(\cdot)$ respectively.
- Then, the **Nash Bargaining Solution** $p^\star$ is given by the p that solves the following problem:

$$\max_p \ (v_b - U_b(p))^\gamma (U_s(p) - v_s)^{1-\gamma}$$

Nash Bargaining Example

- Suppose $U_b(p) = \alpha p$ while $U_s(p) = \theta p$. Then the Nash Bargaining problem is:

$$\max_p \ (v_b - \alpha p)^{\gamma} (\theta p - v_s)^{1-\gamma}$$

- The FOC of this problem is:

$$-\alpha\gamma (v_b - \alpha p)^{\gamma-1} (\theta p - v_s)^{1-\gamma} + \theta(1-\gamma)(v_b - \alpha p)^{\gamma} (\theta p - v_s)^{-\gamma} = 0.$$

- Simplifying this expression, it becomes:

$$\theta(1-\gamma)(v_b - \alpha p) = \alpha\gamma(\theta p - v_s),$$

which implies that:

$$p^{\star} = \frac{\theta(1-\gamma)v_b + \alpha\gamma v_s}{\alpha\theta} = (1-\gamma)\left[\frac{v_b}{\alpha}\right] + \gamma\left[\frac{v_s}{\theta}\right].$$

Wage Bargaining Example 1

- Imagine a worker and an employer recently met and are bargaining over the wage the worker will receive. Both live for an infinite number of periods.

- If the employer hires the worker at wage w, then every period she produces one unit of a consumption good that the firm sells at price p. The employer has an utility function given by:

$$\sum_{t=0}^{\infty} \beta_e^t \pi_t,$$

where β_e measures the impatience level of the firm and π_t are its period profits.

Wage Bargaining Example 1

- If the employer works then she receives a wage w and if she does not work, the government pays her z as an unemployment insurance. Suppose that the government insurance is only available for one period. The utility of the worker is:

$$\sum_{t=0}^{\infty} \beta_w^t x_t,$$

where x_t is the income the worker receives at period t and β_w is her impatience parameter.

- Finally, assume that once both agree on a wage, the job lasts forever and the worker cannot be fired.

- What is the Nash bargaining solution wage? What happens to this wage as the worker is more impatient? Intuition?

Wage Bargaining Example 1

- First, let us notice that $\pi_t = p - w$, which implies that (if an agreement is reached) the utility of the firm is given by $(p - w)/(1 - \beta_e)$.
- Now, if the worker is hired at wage w she will receive that income for the rest of her life, receiving a utility of $w/(1 - \beta_w)$.
  - If she is not hired, she receives $x_0 = z$ and $x_t = 0$ for $t \geq 1$, hence she gets a utility of z.
- Then the Nash bargaining solution problem is:

$$\max_w \quad \left(\frac{p - w}{1 - \beta_e}\right)^{\gamma} \left(\frac{w}{1 - \beta_w} - z\right)^{1-\gamma}.$$

- The solution is then (please verify it):

$$w^{\star} = (1 - \gamma)p + \gamma(1 - \beta_w)z.$$

- Notice that this wage is decreasing in β hence the more impatient the worker is (smaller β) both agree on a hire wage (intuition?).

Wage Bargaining Example 2

- Now let us turn to a more interesting example which will be useful in the future.
- In the same context as the problem before, now suppose that there is a probability λ that the worker gets fired.
  - If the worker gets fired, she gets the unemployment insurance but will not find another job.
  - If the firm fires the worker then with probability η it finds a replacement.
  - We assume that the firm must pay $\overline{w}$ to this new worker. This new worker **cannot** get fired.
- What is the Nash Bargaining solution to this problem? What happens to the wage as λ increases? As η goes to zero? Intuition?

Wage Bargaining Example 2

- To solve this problem, it will be convenient to think about each of the firm and worker's problem in recursive form (believe me, it is easier this way).
- Let $V^e(w)$ denote the value function of the employer/firm if she pays a wage of w.
  - If the firm hires the worker, it gets a utility of $p - w$ in the current period.
  - Now in the future one of two things can occur:
    - ★ The worker remains as an employee (with probability $1 - \lambda$), in which case the firm gets a utility of $\beta_e(1 - \lambda)V^e(w)$.
    - ★ The worker is fired. In such case with probability η the firm hires a new worker, in that case the firm gets a utility of $\lambda\eta\beta_e(p - \overline{w})$.
- Then the firm's value function must satisfy:

$$V^e(w) = p - w + \beta_e(1 - \lambda)V^e(w) + \beta_e\lambda\eta(p - \overline{w}),$$

$$V^e(w) = \frac{p - w + \beta_e\lambda\eta(p - \overline{w})}{1 - \beta_e(1 - \lambda)}$$

Wage Bargaining Example 2

- Let $V^w(w)$ be the worker's value function whenever she gets a wage of w.
  - If she gets hired, then her period utility is given by w.
  - In the future, one of two things can occur:
    - ★ She can continue to work, in which case she receives $\beta_w(1-\lambda)V^w(w)$ as utility.
    - ★ She could get fired, in which case her income becomes z, giving her a utility of $\beta_w \lambda z$.
- Then, the worker's value function is:

$$V^w(w) = w + \beta_w(1-\lambda)V^w(w) + \beta_w \lambda z,$$

$$V^w(w) = \frac{w + \beta_w \lambda z}{1 - \beta_w(1-\lambda)}.$$

Wage Bargaining Example 2

- To close up the problem we need to think: what is the outside option of each agent?

- Notice that V^e, V^w are the utilities for each agent **conditional on agreement**.

- For the firm, if an agreement is not reached, then it can look for another worker.
    - This other worker is found with probability η and must be payed $\overline{w}$.
    - Hence the value of the outside option for the firm is $\eta(p - \overline{w})$.

- On the other hand, if the worker is not hired, she receives an unemployment insurance of z. This is the value of her outside option.

Wage Bargaining Example 2

- Then, the Nash bargaining problem is:

$$\max_w \quad \left(\frac{p - w + \beta_e \lambda \eta (p - \overline{w})}{1 - \beta_e (1 - \lambda)} - \eta (p - \overline{w}) \right)^{\gamma} \left(\frac{w + \beta_w \lambda z}{1 - \beta_w (1 - \lambda)} - z \right)^{1-\gamma}$$

- Notice that this problem can be simplified to (please verify it):

$$\max_w \quad \left(p - w - (1 - \beta_e)\eta(p - \overline{w})\right)^{\gamma} \left(w - (1 - \beta_w)z\right)^{1-\gamma}.$$

- Then the Nash bargaining solution is given by (please verify it):

$$w^{\star} = \gamma(1 - \beta_w)z + (1 - \gamma)(p - \eta(1 - \beta_e)(p - \overline{w})).$$

Search and Matching

- Trade in the labor market has two main characteristics:
  - It takes effort and time.
  - It is uncoordinated (meaning not necessarily workers and firms look for each other at the same time).
- Hence, the households looking for a job (unemployed) need to find (by divine coincidence) a firm that happens to be looking to fill a vacant they have.
- How do we model this process?
  - With a **Matching Function**.
  - This tells us the number of worker-employee relationships that could potentially (depending on the wage bargaining) take place if u percent of households are looking for a job and v percent of all available jobs are vacant.
  - Throughout the course, we will assume that the matching function is given by:

$$m(u, v) = u^{\eta} v^{1-\eta}.$$

Search and Matching: Market Tightness

- For example, suppose $\eta = 1/3$. Then if there are $u = 20\%$ of households looking for a job and there are $v = 10\%$ vacant jobs in this economy. Then the number of matches (you can think a match as an interview, because even if a match occurs they must bargain the wage):

$$m(0.2, 0.1) = (0.2)^{1/3}(0.1)^{2/3} = 0.12.$$

- We define **Market Tightness** as $\theta = v/u$, the proportion of vacants relative to unemployed households.
- Let us define $q(\theta)$ as the proportion of matches relative to vacancies:

$$q(\theta) = \frac{u^{\eta} v^{1-\eta}}{v} = \left[\frac{u}{v}\right]^{\eta} = \left[\frac{1}{\theta}\right]^{\eta}$$

- The proportion of matches relative to unemployed households is then:

$$\frac{u^{\eta} v^{1-\eta}}{u} = \left[\frac{v}{u}\right]^{1-\eta} = \theta^{1-\eta} = \theta q(\theta).$$

Search and Matching: Duration of Unemployment

- How long is (on average) a person unemployed? What determines this duration?
  - In the context of a matching model, we can express unemployment duration as a function of market tightness.
  - Remember $\theta q(\theta)$ is the proportion of matches relative to unemployment.
  - If we consider unemployment duration to be random (which given this context it is) then it is a Poisson variable with parameter $\theta q(\theta)$.
  - Hence the mean duration of unemployment is given by $1/\theta q(\theta) = 1/\theta^{1-\eta}$.
  - What happens with unemployment duration if a market is tighter?
- Similarly, the duration of a vacancy follows a Poisson process with parameter $q(\theta)$ implying that the mean duration of a vacancy is $1/q(\theta) = \theta^{\eta}$.

Search and Matching: Externalities

- What happens to the duration of unemployment as more and more people search for a job but the number of vacancies does not change?
- The average unemployment duration increases!
  - This phenomena is called **congestion externality**.
  - This **does not occur in a model without search**.
  - For example, in the classical model, prices are such that the amount of labor households want to supply is exactly equal to the number of "vacancies". Hence workers do not compete in equilibrium for a job.
  - With search, since it is costly for both to find a match, workers are competing with each other to find a match "first".
  - This suggests a preliminary answer to the question: Why does unemployment fluctuate with the business cycle?
    - ⋆ In a recession, vacancies tend to not increase but the number of unemployed increases, which in turn tightens the market and, hence, unemployment becomes more persistent!

Pissarides' Model

- We now turn to analyze a model that puts together all the pieces we just discussed, first explored by Pissarides (1985).
- Time is discrete $t = 0, 1, 2, \ldots$ and the only market there exists in the economy is the labor market (no capital, no consumption, etc...).
- There is a large number of **identical** workers who have utility:

$$\sum_{t=0}^{\infty} \beta^t x_t,$$

  where x_t is their income at period t.
- Firms are atomistic and **can only hire one worker** (i.e. there are as much firms as employed workers).
  - Firms sell their output in a competitive market that pays price p for it.
  - One worker produces one unit of the good the firm sells per period.
  - Firms must pay a cost $c > 0$ to open a vacancy (which is equivalent to say they must pay a cost c to operate in the market).
  - Firms also discount the future with parameter β.

Pissarides' Model

- Basically, the model works as follows:

  1. At every period unemployed workers and incumbent firms have to search for each other, and matches occur according to $m(u, v) = u^{\eta} v^{1-\eta}$.
  2. Once a match takes place, the worker and firm must agree on the wage. The wage they agree upon is equal to the Nash bargaining solution.
  3. Jobs can be destroyed with probability λ each period. Importantly, if the worker is fired, we assume that the firm closes (and hence has zero profits).
  4. Firms can decide to enter to the market at every time. If they had positive profits, an infinite number of incumbent firms will enter the market. Hence, the model pushes profits to be equal to zero in equilibrium.
  5. Instead of thinking in the proportion of unemployed and vacancies, we will deal with the number of unemployed (u) and **market tightness** (θ).

Evolution of Unemployment

- Now that the model is dynamic, we need to think about how unemployment evolves:
  - Every period $u_t^\eta v_t^{1-\eta} = u_t \theta_t q(\theta_t) = u_t \theta_t^{1-\eta}$ matches occur.
  - Hence, $u_t \theta_t^{1-\eta}$ of the unemployed households at period t will no longer be unemployed at period $t+1$.
  - On the other hand, of the employed workers a proportion λ of them will loose their jobs.
  - Then, $\lambda(1-u_t)$ workers will be unemployed at $t+1$.
- This suggests that the evolution of unemployment $\Delta u_t = u_{t+1} - u_t$ is given by:

$$\Delta u_t = \lambda(1-u_t) - u_t \theta_t^{1-\eta}.$$

Beveridge Curve

- To simplify our lives, we will be analyzing only steady state dynamics.
- In a steady state, unemployment is constant (this does not mean that the same workers are unemployed all the time) and hence:

$$\Delta u_t = 0 = \lambda(1 - u_t) - u_t \theta_t^{1-\eta},$$

which then gives us the **Beveridge Curve**:

$$u_t = \frac{\lambda}{\lambda + \theta_t^{1-\eta}}.$$

- Which is a downward slope curve as a function of θ.

Beveridge Curve in the Data

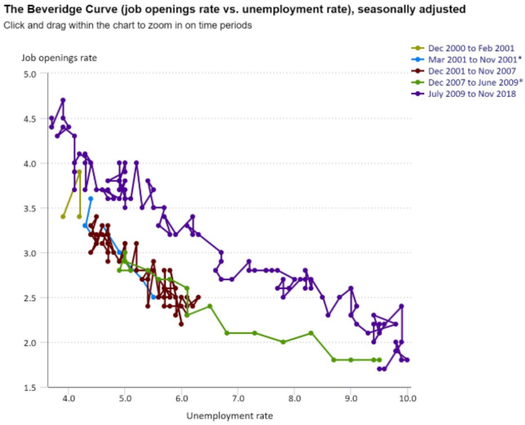

Figure: Beveridge Curve

Closing Up the Model

- Once a match takes place, the future worker and firm negotiate on the wage.
- The Nash bargaining solution is the solution to:

$$\max_{w_t} \left(V^e(w) - \theta_t^{1-\eta} c\right)^{\gamma} \left(V^w(w) - z\right)^{1-\gamma},$$

where:

$$V^e(w) = p - w + \beta\left[(1-\lambda)V^e(w) + \lambda 0\right] \quad \Rightarrow \quad V^e(w) = \frac{p - w}{1 - \beta(1-\lambda)},$$

$$V^w(w) = w + \beta\left[(1-\lambda)V^w(w) + \lambda z\right] \quad \Rightarrow \quad V^w(w) = \frac{w + \beta\lambda z}{1 - \beta(1-\lambda)}.$$

Closing Up the Model

- The Nash bargaining solution is:

$$w_t = (1-\gamma)(1-\beta(1-\lambda))z + \gamma(p - (1-\beta(1-\lambda))\theta_t^{1-\eta}c).$$

- How do we determine market tightness? As we mentioned before, incumbent firms need to have zero profits. Then:

$$V^e(w^\star) - (1-\beta)c\theta_t^{1-\eta} = p - w_t - (1-\beta(1-\lambda))c\theta_t^{1-\eta} = 0$$

Closing Up the Model

- Then, the following are three equations on three unknowns (unemployment, wages, and market tightness):

$$u_t = \frac{\lambda}{\lambda + \theta_t^{1-\eta}},$$

$$w_t = (1-\gamma)(1-\beta(1-\lambda))z + \gamma(p - (1-\beta(1-\lambda))\theta_t^{1-\eta}c),$$

$$p - w_t - (1-(1-\beta(1-\lambda)))c\theta_t^{1-\eta} = 0,$$

for which we can solve the model.

Beveridge Diagram

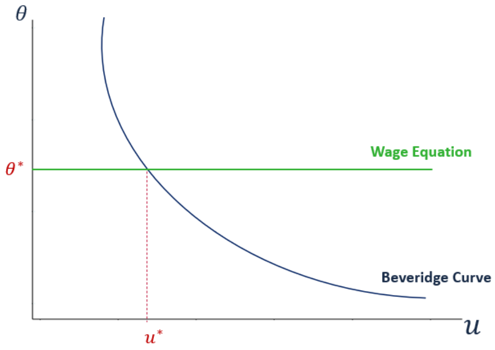

Figure: Beveridge Curve

Efficiency?

- As we highlighted before, there are externalities in this model (why?). This suggests that the equilibrium outcome might not be efficient (why?).
- Remember, to discuss efficiency we need to solve the Social Planner's Problem.
  - What does the social planner consider to solve her problem?
  - Recall that **the social planner never considers prices, only feasibility**.
  - The social planner needs to consider there are interactions between workers and firms. Hence, the SPP includes the Beveridge Curve.
  - What does the Social Planner seeks to maximize? In this model both workers and firms only value wealth.

Efficiency?

- The SPP is:

$$\max_{u,\theta} \sum_{t=0}^{\infty} \beta^t \left[p(1-u_t) + zu_t - c\theta_t^{1-\eta} u_t \right] \text{ subject to}$$

$$u_t = \frac{\lambda}{\lambda + \theta^{1-\eta}}.$$

- **Hossio's Rule (1990)**: The equilibrium outcome is efficient if and only if $\eta = \gamma$.
  - If $\eta > \gamma$ then unemployment in equilibrium will be below its social optimum.
  - Intuition?

Other Models on Unemployment

- Burdett and Mortensen (1998) is another seminal paper in the macro literature about unemployment. The main differences with Pissarides' model are:
  - There is **Wage Posting** instead of bargaining: firms do a take-it-or-leave-it wage offer and if workers do not accept, they need to search for another firm.
  - Workers follow a **reservation wage strategy**.
- Main insights:
  - This model accounts for **wage dispersion**: two workers that do exactly the same job can be paid different wages.
  - Workers perform optimally on-the-job search.
  - Why is there unemployment in this model? Because some firms post wages that are below the worker's reservation wage, hence, they found it optimally to continue searching for a job (and meanwhile, they are unemployed).

Other Models on Unemployment

- There are also models, like the one in Moen (1997), of **Competitive Search**.
  - The main ingredient of a competitive search model is a **Lucas Islands** structure.
- Lucas Islands: Imagine a world in which there are a large number of small (atomistic) islands.
  - In each island there is a single firm that produces a good that can be substituted by the one produced in another island.
  - This firm postes a wage.
  - Consumers search for the “right” island in equilibrium.

Thinking Critically About The Model

- What are the roles of search and unemployment frictions?
- What are the main assumptions of Pissarides' model?
  - Do we believe these assumptions are realistic?
  - Which one would you criticize the most?
- What are the limitations of search models?
  - Give a concrete example of a situation where using a search would not be useful.

Outline

Monetary and Fiscal Policy

- We will wrap up this course with a discussion on how to think about Monetary and Fiscal Policy.
- The first takeaway of this discussion is that **one cannot discuss either without a model in mind**.
  - We will see that the relevance of monetary and fiscal policies will vary as we change assumptions.
  - Universal truth?
- We will mainly focus on the following topics:
  1. Money and its (potential) neutrality.
  2. What gives money its value?
  3. Ricardian Equivalence and its relevance.
  4. Interaction between monetary and fiscal policy.
  5. Monetary and Fiscal Policy in Mexico.

Money and Neutrality: Questions

- Is it worth studying models with money?
- When is money relevant/or not?
- What is the role of monetary policy then?
- Up to what extent is money necessary to explain some phenomenon we observe in the data?

Money and Neutrality

- Why do we think that money is neutral?
  - Neutrality of Money: No real variable is directly affected by the amount of money in the economy.
  - It is an implication of the Classical Growth Model!
  - Why?
- It is a known fact that you need to assume at least one **friction** in your model in order to be able to talk about the non-neutrality of money.
- We have already saw a model in which money is relevant: **OLG**!
  - Money is a mechanism that allows generations to induce trade between old and young.
  - But is it neutral? If we change the size of m do we affect real variables like savings, or consumption?
  - Money is **not** neutral in OLG models. Remember the offer curves.

Money and Neutrality: NKM

- Is money neutral within a NKM?
- No! The main (intuituve) reasons of this answer are the following:
  - In NKM some prices are sticky.
  - Remember the money/output identity:

$$M_t v_t = P_t Y_t,$$

    where v_t is "money velocity".
  - When money increases, Y_t and hence C_t will likely increase.
  - In a context of **flexible prices** P_t adjusts and hence, the increase in M_t dilutes.
  - What about if prices are sticky?
  - Some firms would love to adjust p_i but they can't. Hence, their c_i will effectively change due to this money increase!
  - Then, money affects consumption and output!

Money and Neutrality: Menu Cost Model

- Let us consider a Menu Cost Model, where now households solve:

$$\max \quad \left[\sum_{i=1}^{N} c_i^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \quad \text{subject to}$$

$$\sum_{i=1}^{N} p_i c_i = M,$$

  where M is the amount of money in the economy (we are assuming a constant money velocity equal to 1).

- The FOCs of this problem imply that the demand the consumer has for each variety is given by:

$$c_i = \left[\frac{p_i}{\mathcal{P}}\right]^{-\sigma} \frac{M}{\mathcal{P}},$$

Menu Cost Model: Firm's Problem

- What do firms consider to set their prices?
- Now in a period, profits are given by:

$$\pi(z_i, \mathcal{P}, p_i, M) = z_i \left[\frac{p_i^{1-\sigma}}{\mathcal{P}^{-\sigma}}\right] \frac{I}{\mathcal{P}} - m_i \left[\frac{p_i}{\mathcal{P}}\right]^{-\sigma} \frac{M}{\mathcal{P}}.$$

- Hence, the state variables for a firm are now:

1. Their current productivity shock z_i.
2. The aggregate price index $\mathcal{P}$.
3. The price they set on the previous period p_i^-.
4. **The amount of money in the economy** M.

Menu Cost Model: Firm's Problem

- Firm i then solves:

$$V_i(z_i, \mathcal{P}, p_i^-, M) =$$

$$\max_{p_i} \left\{ \pi(z_i, \mathcal{P}, p_i, M) - \chi \, 1_{p_i \neq p_i^-} + \beta \;\; \mathbb{E}_{z_i', \mathcal{P}', M'} \left[V_i(z_i', \mathcal{P}', p_i, M') \right] \right\},$$

  where $1_{p_i \neq p_i^-}$ is a function equal to one only if p_i (the price that the firm is currently choosing) is different from p_i^-.

- Now, the firm must "predict" the amount of money there will be in the economy.
  - Why?
  - More money translates into a higher demand! (why?)
  - Again, the **Rational Expectations** problem appears (where?).

Menu Cost Model: Monetary Policy Shocks

- We will assume that money evolves according to the following **Monetary Policy Rule**:

$$M_t = (1-\rho)M_0 + \rho M_{t-1} + \epsilon_t^M,$$

where ϵ_t^M is known as **Monetary Policy Shock**.

- Why does this captures a monetary policy shock?
  - The $(1-\rho)M_0 + \rho M_{t-1}$ component, can be easily predicted by everyone at the economy.
  - However ϵ_t^M is a "random walk" variable and hence, by definition, cannot be predicted.
  - What is in ϵ_t^M?
  - Interest rate announcements, unconventional monetary policy rules, etc...

Monetary Policy Relevance

- Now, I present a simulation of the menu cost model that incorporates money and I will answer (highlight) the following:

  1. What is the effect on the price-setting behavior of firms of a one standard deviation monetary policy shock?
  2. What is the effect of this shock on consumption? Output?
  3. If we let money fluctuate according to $M_t = (1-\rho)M_0 + \rho M_{t-1} + \epsilon_t^M$, how much of the output/consumption variance is due to movements in money?

Non-Neutrality in NKM

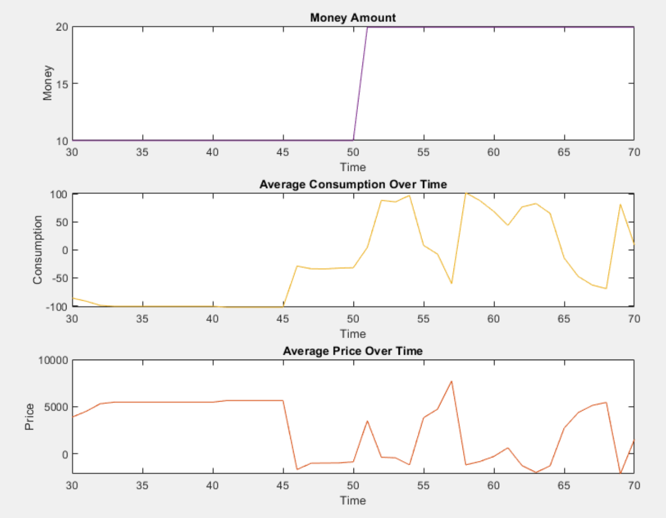

Figure: Evolution of Aggregates with Money

Non-Neutrality in NKM

- Here are some statistics that will allow us to understand the effects of a monetary shock.

Average Price After Shock/ Before Shock	0.8840
Average Productivity Shock After Shock/ Before Shock	1.0029
Average Consumption After Shock/ Before Shock	2.7146
Average Utility After Shock/ Before Shock	2.1957
Average Duration of Prices After Shock / Before Shock	1.0213

Table: Statistics of the Simulated Menu Cost Model with Money

How Relevant is Monetary Policy?

- To assess the relevance of monetary policy (as described for this model), I present two sets of estimations: one where money fluctuates according to the Monetary Policy Rule $M_t = (1-\rho)M_0 + \rho M_{t-1} + \epsilon_t^M$, and another one where $M = 10$ (the implied mean of the rule).
- How much do prices and consumption/output variation can be explained by the presence of money?

Consumption Variance with/Without Money Fluctuations	5.0866 %
Prices Variance with/Without Money Fluctuations	3.1296 %

Table: Aggregates in the Model With/Without Money Fluctuations

Non-Neutrality in NKM

Figure: Evolution of Aggregates in the Model with Money Fluctuations

Non-Neutrality in NKM

Figure: Evolution of Aggregates in the Model with Money Fluctuations

How Relevant is Monetary Policy?

- The previous example illustrates that by adding money into our model, we are able to explain between 3-5% more of the business cycle fluctuations.
- This is not exclusive of Neo-Keynesian models, we would get a similar result if we considered models with money in the utility function or cash in advance.
- Hence, a natural question arises: do we really need money in order to study macroeconomic fluctuations?
  - Classical view, based on Woodford (1998): Medium-of-exchange considerations are irrelevant for monetary transmission in modern high-velocity credit economies.
    - This means that models with/without money generate basically the same results.

How Relevant is Monetary Policy?

- Lagos and Zhang (2021): Woodford's approach is correct in economies in which there only exist pricing frictions.
- Economies in which there are credit/financial frictions money and its use play a crucial role.
- Why?
  - Money is an (imperfect) substitute of bonds: both allow agents to save for future consumption although at different interest rates.
  - In a model with credit/financial frictions, the classic Euler equation relating prices and interest rates does not hold.
  - Hence, changing the amount of money in the economy changes the price of money, but does not alter the interest rate, then impacting the substitution between money and bonds.

What Gives Money its Value?

- Let us assume that we live in an economy in which money is relevant.
- Where does the value of money come from?
- Several theories:
  1. Money is a medium of exchange, i.e., fiat money as a substitute of barter (Kiyotaki and Wright, 1993).
  2. Money is memory (Kocherlakota, 1997).
  3. Money is a stock (Cochrane, 2005).
  4. Modern Monetary Theory (Kelton, 2021).

Monetary Policy: Other Financial Institutions

- Besides Central Banks, there are other financial institutions that are relevant for monetary policy transmission.
- Let us briefly discuss the role of private banks.
- Why do they matter?
  - Keynesian answer: because they amplify the monetary base and they alter the velocity of money.
  - Gorton, Holmstrom and Ordonez (2017): Private Banks are mainly relevant because they are secret keepers.

Monetary Policy: Banks As Secret Keepers

- Imagine an economy in which start-ups need credit to fund their projects.
  - A new project p_i can be either successful and generate a revenue for the investor of Y_i or fail and give 0 to the investor.
  - Let α_i be the probability that p_i is successful.
  - Let us imagine that only the start-up and the investor know α_i, an outside observer cannot know exactly the probability that the project succeeds.
- What would happen in a world in which there are no private banks?
  - Consumers/households would be the investors in this economy.
  - But then, since consumers at the end of the day only value consumption, they would only invest in projects that have a positive expected value.
  - Many projects that are riskier would not be funded.

Monetary Policy: Banks As Secret Keepers

- What is the role of a private bank then?
  - Private banks have a large amount of capital to invest, funded by private depositors.
  - Hence, they can take bigger risks and invest in projects that a normal household would not invest.
  - They diversify their portfolio, but, it is in the best interest of the bank that this is kept a secret from depositors.
  - Why? If people knew exactly the risk the bank is taking, they would not deposit their money in it.
- This is another potential explanation of why, when people stops believing in financial institutions, banks run occurs as in Diamond and Dybvig (1983).

Fiscal Policy: Ricardian Equivalence

- How should a government finance a stream of expenditures?
- Two possibilities:
  1. Taxes.
  2. Issue government debt.
- Which are the consequences on welfare, consumption, prices of each option?
- Is there a some-what fiscal policy neutrality?
- **Ricardian Equivalence**: Under certain conditions (which we shall explore) it makes absolutely no difference!

Ricardian Equivalence

- When shall we expect there is Ricardian Equivalence?

- Whenever there are complete markets and no pricing frictions!
  - That is, in the Classical Growth Model Ricardian equivalence holds.
  - Why?
  - Think of an Arrow-Debreu setting.
  - Consumers only care about satisfying one inter-temporal budget constraint.
  - Hence, as long as the tax and debt sequence leaves that budget set untouched, the optimal consumption, prices, etc... will be exactly the same!

Ricardian Equivalence Theorem

- Take as given an government expenditures sequence $\{g_t\}_{t=0}^{\infty}$ and an initial debt D_0.
- Suppose we have an Arrow-Debreu equilibrium $\{\hat{c}_t, \hat{p}_t\}_{t=0}^{\infty}$ for which there is a tax sequence $\{\tau_t\}_{t=0}^{\infty}$ that satisfies the government budget constraint:

$$\sum_{t=0}^{\infty} \hat{p}_t g_t \leq \sum_{t=0}^{\infty} \hat{p}_t \tau_t.$$

- Let $\{\tilde{\tau}_t\}_{t=0}^{\infty}$ be an arbitrary tax sequence satisfying:

$$\sum_{t=0}^{\infty} \hat{p}_t \tilde{\tau}_t = \sum_{t=0}^{\infty} \hat{p}_t \tau_t.$$

- Then the sequence $\{\hat{c}_t, \hat{p}_t\}_{t=0}^{\infty}$ also constitute an Arrow-Debreu equilibrium associated to $\{\tilde{\tau}_t\}_{t=0}^{\infty}$.

Ricardian Equivalence Theorem: Some Remarks

- For this result to hold, the sequence of government expenditures must remain fixed.
- Ricardian equivalence **does not imply that the timing of expenditures is irrelevant**, only how they are taxed to consumers.
- Importance of complete markets and absence of frictions.

Ricardian Equivalence Outside the Classical Model

- Should we expect Ricardian equivalence to hold if we explore other models?
- It will probably not hold. Let us consider the OLG case:
  - The main reason why Ricardian equivalence holds, is because by changing the tax sequence, consumers are interchanging future/present consumption is such a way they remain indifferent.
  - Can this mechanism occur in an OLG model?
  - No! Why? Agents in this model die!
  - Hence, if the government is thinking of taxing the old vs. the young, they will generate an effect on consumption and savings of either one.

Fiscal and Monetary Policy Interactions

- How does fiscal policy affect monetary policy and vice-versa?
- In the literature, there are two types of regimes that are usually explored:
  1. **Fiscal Dominance**: A situation in which the Monetary Authority targets government deficit and debt related variables and, hence, its influence on other economic outcomes such as inflation and output is limited.
     - Cagan (1954), Blanchard (2004), Sargent et al. (2009).
  2. **Monetary Dominance**: Also known as Monetary Independence. A regime in which the Monetary Authority can focus on inflation, output, or other objectives not directly related to fiscal variables.
     - Leeper (2001), Woodford (2001), Sims (2006).
- Nowadays most of the countries in the world are in a Monetary Dominance regime.
  - This has been a consequence of a wide variety of experiences suffered by countries during the post-war period where high debt levels led to hyperinflations and recessions.

Fiscal and Monetary Policy Interactions

- How disconnected can fiscal and monetary policy be even in a context of Monetary dominance?
  - Not that much!
  - Sargent (2018): **In Latin America, inflation is always and everywhere a fiscal phenomenon**.
  - Kocherlakota (2012): Agents are rational, and hence, if they observe an imprudent fiscal stance they will incorporate into their behavior the possibility of default, a situation in which the Monetary Authority will have to intervene and aid.
  - Carstens (2005): Fiscal imbalances should not be treated as lightly, they affect the effectiveness of monetary policy.
  - Fiscal Theory of the Price Level: Christiano and Fitzgerald (2000), Sims (2016).

Fiscal and Monetary Policy Interactions

- Just to fix ideas lets analyze a more concrete example.
- Consider an economy in which the government's constraint is given by:

$$\tau_t^n w_t n_t + \tau_t^k r_t k_t + D_{t+1} = g_t + (1 + r_t) D_t + T_t,$$

  where τ^n is a labor income tax, τ^k is a capital income tax, g are government expenditures, T are transfers, and D is government debt.

- What happens when the Monetary Authority implements a certain policy? Let us think that they use as instrument the interest rate (which, remember, is inversely related to the money amount in the economy).
  - **Restrictive Monetary Policy**: This increases interest rates.
    - Higher interest rates implies a higher financial cost of debt $(1 + r_t)D_t$.
    - What if the government is in a situation where D_t is considerably high?
    - Ambiguous General Equilibrium effect on capital related income.

Fiscal and Monetary Policy Interactions

- Why do we care so much about government debt?
  - Mainly, because a higher debt (which eventually has to be repaid) usually implies higher taxes.
  - But also, who is entitled to that debt? Consumers!
  - The government borrows from consumers' savings (there is no one else to borrow from).
  - Then, Monetary Policy can have distributive effects on consumers since it affects their savings!
    - ⋆ Key insight of Kaplan et al. (2018) **Monetary Policy According to HANK**.
  - Relevance of Ricardian equivalence in this discussion.
- So, why does Carstens (2005) says that Fiscal Policy can pose as a threat to Monetary Policy?
  - Imagine a scenario in which D_t is very high.
  - Agents know that the Monetary Authority simply **cannot increase interest rates**.
  - Hence, they anticipate this (rational expectations) modifying their behavior.

Fiscal and Monetary Policy Interactions in Mexico

- As documented by Cardenas (2015), before 1994 there existed a fiscal dominance regime in Mexico.
- Fiscal deficit relative to output was one of the main forces that pushed prices upward (together with the exchange rate).
  - After the 1982 fiscal crisis and the collapse of oil prices in 1986, fiscal deficit relative to output reached levels of 14% of GDP.
  - The deficit is important since, in fiscal dominance regimes, it is usually translated (via the Central Bank) into seigniorage and hence, inflation.
- After 1994, year in which Mexico's Central Bank achieved its autonomy, inflation has achieved historical lows (2%), and (as some articles suggest) has lost its high correlation with fiscal policy.
- Nevertheless, as documented by Lopez-Martín et al. (2018) inflation expectations are a channel through which the fiscal policy can (indirectly) affect prices.
  - Importance of a prudent Fiscal Authority.

Fiscal and Monetary Policy Interactions in Mexico

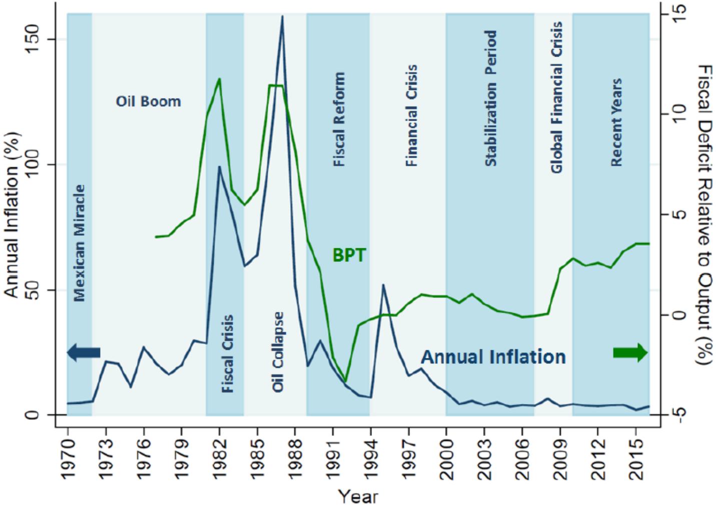

Figure: Inflation and Fiscal Deficit Dynamics in Mexico.